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A Helical Wave Guide II

by

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CONTENTS

Note

A paper with the same title was submitted to the Watson Laboratories 1 January 1947. This was special report No. 170-2. These two papers differ fundamentally in the boundary conditions that are assumed. In the earlier report it was assumed that on the surface of the guide the component of the magnetic field (as well as that of the electric field) in the helical direction is zero. In the present study we assume that the component of the magnetic field in the helical direction is continuous as we pass through the surface of the guide, but not necessarily zero on the surface. The newer conditions are not only more realistic physically, but they greatly increase the difficulty of the problem.

Abstract

This is an investigation of a wave guide in the form of an idealized helix which is merely a circular cylindrical surface subjected to the following conditions. An arbitrary helical direction on the surface is chosen and it is assumed that the surface has perfect conductivity in the helical direction and zero conductivity in the direction normal to this on the surface. Consequently on the cylinder the electric field vanishes in the helical direction whereas the magnetic field is continuous in this direction.

For each n(n=0, $\pm$ 1, $\pm$ 2, ---) there is a general solution of the wave equation in cylindrical coordinates involving nth order Bessel functions with n nodes around the circumference. For each such solution there are precisely as many non-attenuated modes as there are real and imaginary solutions in v of the equation

$$\frac{\delta}{\alpha} = -\frac{n}{v^2}\left[1 + \left(\frac{v}{\alpha}\right)^2\right]^{1/2} \pm \sqrt{-\frac{I_n'(v)\,K_n'(v)}{v^2\,I_n(v)\,K_n(v)}}$$

Here $\alpha = \frac{2\pi a}{\lambda_o}$ where λ_o is the free-space wave-length and a is the radius of the guide, and $\delta = \frac{d}{2\pi a}$ where d is the distance between turns of the helix.

It is found that there are no imaginary solutions of the above equation. Each real solution corresponds to a mode with phase velocity along the cylinder smaller than the free-space wave velocity. The paper is therefore concerned with the existence of real solutions for various ranges of values of α and their relations to one another. This problem is discussed rigorously for n = 0 and $|n| \geq 3$. It has not been possible as yet to complete the theoretical discussion for n = $\pm$ 1 or $\pm$ 2, and for these cases graphical evidence is supplied for the conclusions reached.

1. Introduction

In recent years a great deal of research has been done on an ultra-high-frequency amplifier known as the traveling-wave tube.* It has been found that this tube possesses the desirable properties of high gain, broad bandwidth, and low noise level. The essential part of the tube is a helical transmission line which is designed to pass an electromagnetic wave with wave length along the guide, and hence phase velocity along the guide, smaller than that of the free-space wave by a factor of about thirteen. An electron beam with a velocity approximately equal to the phase velocity of the electromagnetic wave is shot down the center of the helical guide. The electromagnetic wave and the electron beam interact, and as a result the wave amplitude is magnified.

This paper consists of a study of the transmission properties of an idealized helical wave guide. The helical wave guide used in the actual construction of the traveling-wave tube is a single wire wound in the shape of a helix. Unfortunately, the propagation properties of the helix itself have thus far eluded investigators in this field. We have considered, instead, an idealized helix obtained by replacing the helix with a cylinder of the same radius which is conducting in only the original helical direction and is non-conducting in the helical direction normal to this.**

This idealization appears to be a very good approximation to the actual helix. It is clear that the current is constrained to travel along a helix having the same pitch in both cases. Thus the boundary conditions are somewhat the same for the two designs. One would expect differences in the field in the immediate neighborhood of the wire or cylinder; these differences should not, however,

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* The traveling-wave tube was developed by a group under the direction of R. Kompfner at Oxform University. The following reports in this field are available:

R. Kompfner, The Traveling Wave Value, Wireless World, vol. 52, 1946, pp. 369-372.

J. R. Pierce, Theory of the Beam-Type Traveling-Wave Tube, Proc. I.R.E., vol.35,1947, pp. 111-123.

J. R. Pierce and L. M. Field, Traveling-Wave Tube, Proc. I.R.E., vol.35,1947, pp. 108-111.

**This same idealization was used by Franz Ollendorff, Die Grundlagen der Hochfrequenztechnik, pp. 79-87, Springer, Berlin, 1926.

appreciably influence the phase velocity of the wave. This is, in fact borne out by experiment. If one applies the theory for the idealized helical wave guide to the helix, one can obtain the observed phase velocity along the guide by using a radius approximately equal to the mean helix-radius. For instance, for a free-space wave length of 82 cm., a spacing between turns of 2 cm., and a helical radius of 3 cm. (inner) and 4.3 cm. (outer), one obtains the experimentally determined phase velocity for the single wire helix by means of the idealized helix theory in which a radius of 4.1 cm. is used. In another instance for a free-space wave length of 10 cm., a spacing between turns of 0.25 cm., and a helical radius of 2.6 mm. (inner) and 3.8 mm. (outer), it is necessary to use in the theory a cylinder radius of 3.1 mm. in order to achieve agreement.

The precise boundary conditions for the idealized helical wave guide which we consider can be formulated as follows. The guide consists of a circular cylinder of zero thickness which extends indefinitely far in both directions. At the surface of the cylinder the tangential electric field is assumed to be continuous. We shall assume perfect conductivity on the cylinder in the helical direction and zero conductivity in the direction normal to the helical direction. Hence on the cylinder the electric field vanishes in the helical direction whereas the magnetic field is continuous in this direction. These boundary conditions lead to the existence of certain normal modes which, in the usual wave guide terminology, are linear combinations of the transverse-electric and transverse-magnetic modes.

For each n ($n = 0, \pm 1, \pm 2, \ldots\ldots$) there is a general solution of the wave equation in cylindrical coordinates involving n^{th} order Bessel functions with n nodes around the circumference. For each such solution there are precisely as many non-attenuated modes as there are real and imaginary solutions in v of the equation

$$\frac{\delta}{\alpha} = - \frac{n}{v^2}\left[1 + \left(\frac{v}{\alpha}\right)^2\right]^{-1/2} \pm \sqrt{- \frac{I_n'(v)\, K_n'(v)}{v^2\, I_n(v)\, K_n(v)}} \quad . \qquad (1.1)$$

Here $\alpha = \frac{2\pi a}{\lambda_0}$ where λ_0 is the free-space wave-length and a is the radius of the cylindrical guide, and $\delta = \frac{d}{2\pi a}$ where d is the distance between turns of the helix. It turns out that there are no imaginary solutions of Eq.(1.1). On the other hand there is always one real solution for $n = 0$. For each $n < 0$ there are either one, two, three, or four real solutions; for sufficiently large $n < 0$ there are exactly

two real solutions. Finally for each $n > 0$ there are either zero, one, or three real solutions; for sufficiently large $n > 0$ there are no real solutions.*

Each real solution of Eq. (1.1) corresponds to a mode with phase velocity along the cylinder smaller than the free-space wave velocity. The real solution modes are therefore the modes of interest in the traveling-wave tube. In order to insure stable operation in this application it is necessary that the mode which interacts with the electron beam be well isolated in phase velocity from any other mode. It is therefore desirable to determine how the different real solutions of Eq. (1.1) are related to one another. The major part of the present work is concerned with precisely this problem.

In order to study the solutions of Eq. (1.1) we have investigated the functions on the right-hand side of this equation. With the plus sign before the radical, the $(n + 1)$st function lies below the nth for all positive and negative n. A graph of these functions for $n = 0, \pm 1, \pm 2, \pm 3$ and $\alpha = 1$ is shown in Fig. 1. When the functions are positive monotonic decreasing as is the case for $n \leq 0$, the solution for the $(n - 1)$st function is a larger v (smaller phase velocity) than the solution for the nth function. For $n = +1$ and $\alpha \leq 1$ there is just one solution; the corresponding value of v is of course even smaller than the $n = 0$ solution. For $n > +1$ and $\alpha < \sqrt{n(n-1)}$ the functions are negative; in this case there is no solution. With the negative sign before the radical, the functions are ordered in precisely the same way. In fact one can obtain the negative-sign functions from the positive-sign functions by replacing n by $-n$ and changing the sign of the function. For $n \leq -2$ and $\alpha \leq 1$ the functions are again positive and monotonic decreasing; a solution will exist if and only if $\frac{\delta}{\alpha} < \frac{n}{2}\left(\frac{1}{\alpha^2} - \frac{1}{n^2 - 1}\right)$; in this case the solution for the $(n - 1)$st function corresponds to a larger v than the solution for the nth function. For $n = -1$ and $\alpha \leq 1$ there is no solution for large $\frac{\delta}{\alpha}$; for sufficiently small $\frac{\delta}{\alpha}$, however, there are two solutions. Finally for $n \geq 0$ the functions are negative-valued so that there is no solution. In general one can achieve the best isolation of the phase velocities of the different modes by choosing α as small as is practicable.

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* The above conclusions for $n = 0$ and for sufficiently large n have been established rigorously. The in-between cases are conjectures based on graphs.

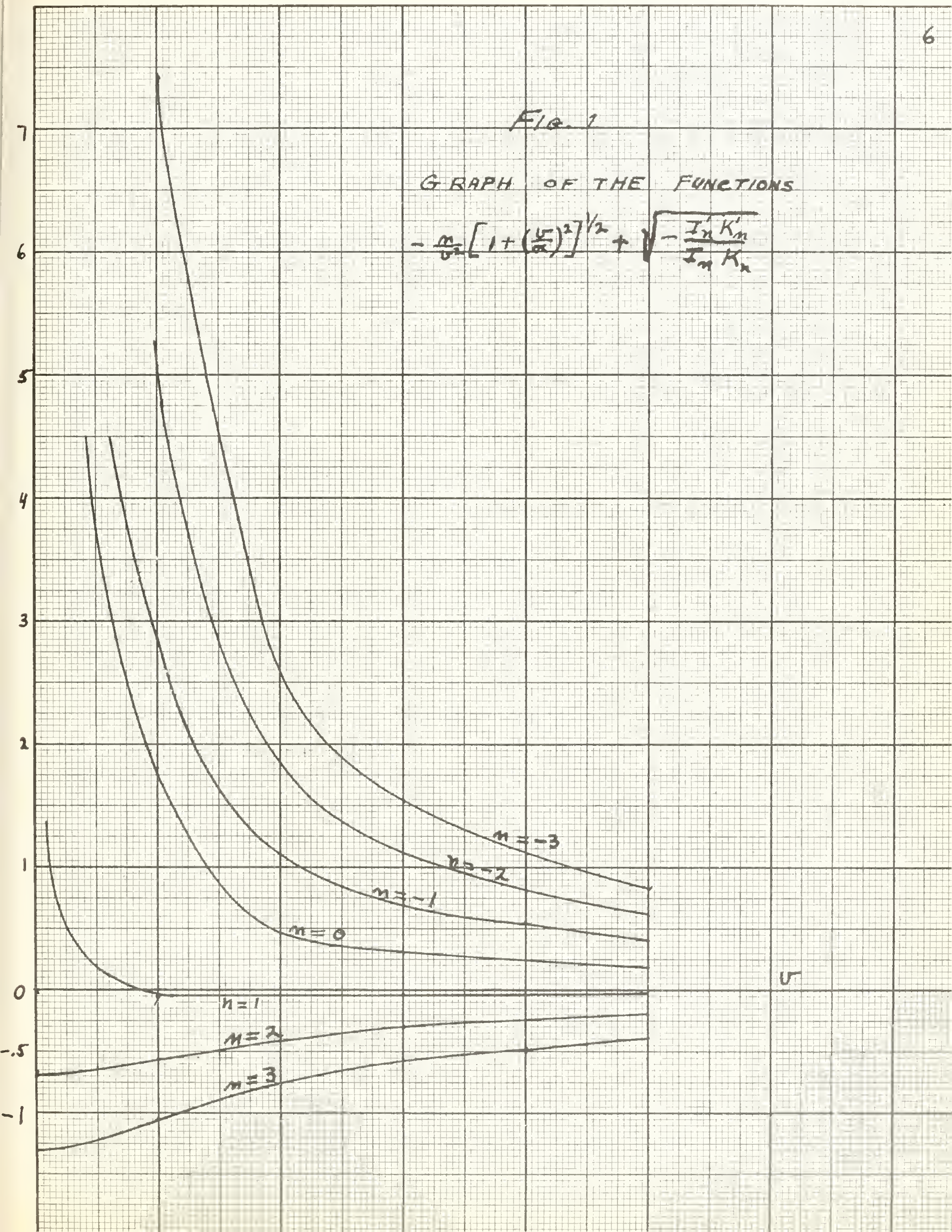

FIG. 1

GRAPH OF THE FUNCTIONS

$$-\frac{n}{v^2}\left[1+\left(\frac{v}{\alpha}\right)^2\right]^{1/2}+\sqrt{-\frac{I_n' K_n'}{I_n K_n}}$$

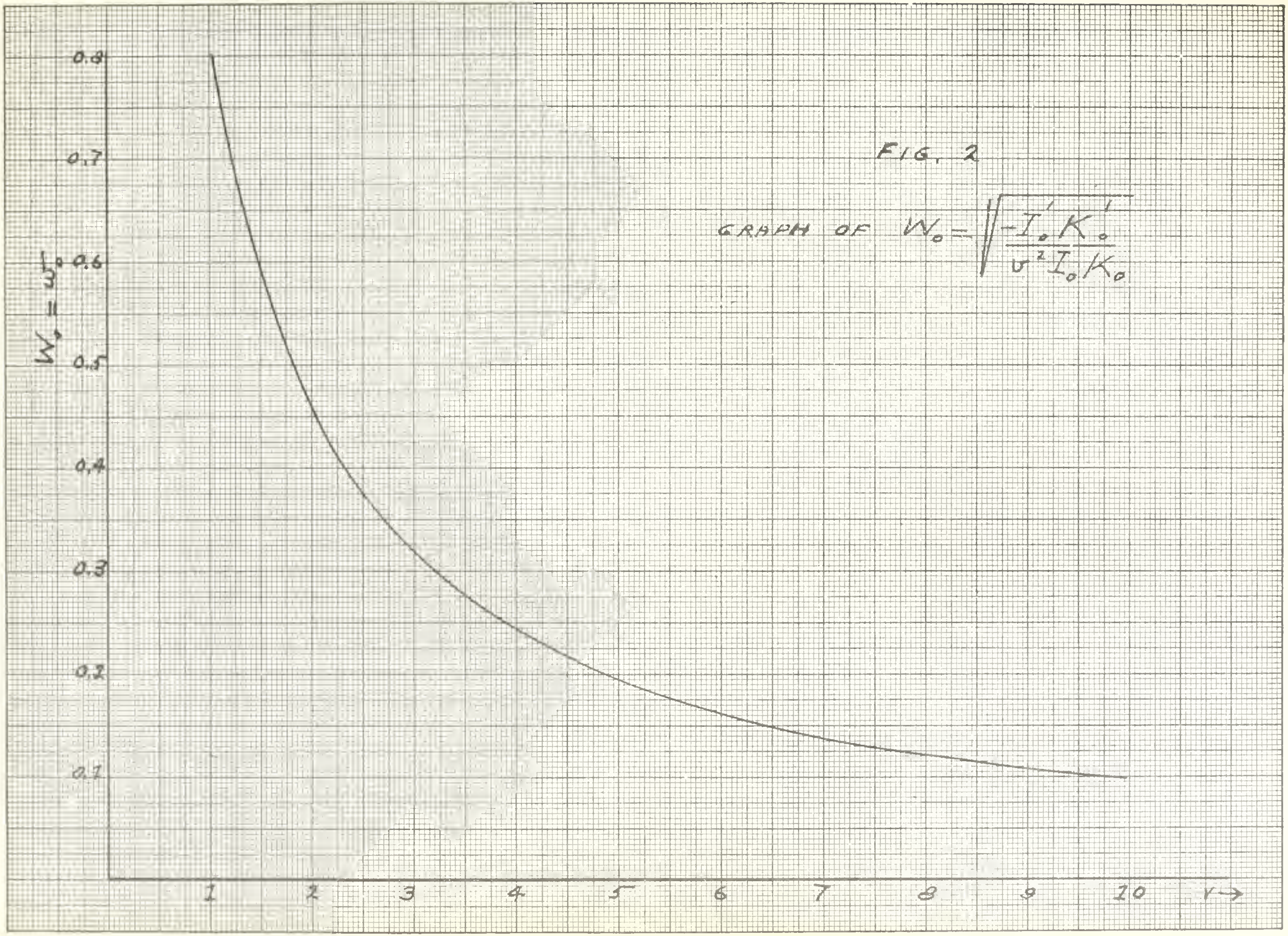

FIG. 2

GRAPH OF $W_0 = \sqrt{\dfrac{-I_0' K_0'}{v^2 I_0 K_0}}$

It is hoped that the above abbreviated description of the functions on the right-hand side of Eq.(1) will suffice to give the reader an idea of the type of results we have sought to obtain. A more complete description of these results will be found in Sec. 10.

In practive, the zero-order mode is used to interact with the electron beam, The phase velocity for this mode is always sandwiched in between that of the (n = 1)-mode and that of the (n = -1)-mode. The solution for the zero-order mode can be read from the graph in Fig. 2. In order to obtain the wave-length along the guide one uses the formulas

$$\lambda_z = \lambda_o \left[1 + \left(\frac{v}{\alpha}\right)^2 \right]^{-1/2} \tag{1.2}$$

For values of $\frac{\delta}{\alpha} < 0.2$, the solution of Eq. (1.1) with n = 0 can be approximated by

$$v_o \simeq \frac{\alpha}{\delta} \tag{1.3}$$

which gives an approximate wave-length of

$$\lambda_z \simeq \lambda_o \frac{\delta}{(1 + \delta^2)^{1/2}} \tag{1.4}$$

This is precisely what one would obtain if the wave followed the helical windings in the cylinder with its free-space velocity.

As we have seen, for sufficiently large negative n, Eq. (1.1) has precisely two solutions for each n. These solutions are close together and for small α give a sort of fine-line structure to the phase velocities. A first order approximation to these solutions is

$$v_n = \frac{\alpha}{\delta}\left(\frac{|n|}{\alpha} \mp 1\right) . \tag{1.5}$$

2. Mathematical Formulation of the Problem.

For a monochromatic source, the electromagnetic field inside of an infinitely long circular cylinder ($r < a$) can be represented in the form*

$$E_r = \sum_{n=-\infty}^{\infty} \left[\frac{i\beta}{\gamma} J_n'(\gamma r) A_n^1 - \frac{\mu\omega n}{\gamma^2 r} J_n(\gamma r) B_n^1 \right] F_n ,$$

(2.1)

$$E_\theta = \sum_{n=-\infty}^{\infty} - \left[\frac{n\beta}{\gamma^2 r} J_n(\gamma r) A_n^1 + \frac{i\mu\omega}{\gamma} J_n'(\gamma r) B_n^1 \right] F_n ;$$

* See J. A. Stratton, Electromagnetic Theory, p. 524, McGraw-Hill, 1941.

$$E_z = \sum_{n=-\infty}^{\infty} J_n(\gamma r)\, A_n^1 F_n \quad ;$$

$$H_r = \sum_{n=-\infty}^{\infty} \left[\frac{nk^2}{\mu\omega\gamma^2 r} J_n(\gamma r)\, A_n^1 + \frac{i\beta}{\gamma} J_n'(\gamma r)\, B_n^1 \right] F_n \quad ,$$

$$H_\theta = \sum_{n=-\infty}^{\infty} \left[\frac{ik^2}{\mu\omega\gamma} J_n'(\gamma r)\, A_n^1 - \frac{n\beta}{\gamma^2 r} J_n(\gamma r)\, B_n^1 \right] F_n \quad , \qquad (2.1)$$

$$H_z = \sum_{n=-\infty}^{\infty} J_n(\gamma r)\, B_n^1\, F_n \quad .$$

In these relations

$$\gamma^2 = k^2 - \beta^2 \,,$$

$$F_n = e^{in\theta + i\beta z - i\omega t} \,, \qquad (2.2)$$

$$k^2 = \varepsilon\mu\omega^2 \,,$$

and the prime denotes differentiation with respect to the argument γr. J_n is the Bessel function of the first kind. The field outside the cylinder (r > a) is the same except that

$$J_n(\gamma r) \longrightarrow H_n^1(\gamma r) \,,$$

$$A_n^1 \longrightarrow A_n^2 \,, \qquad (2.3)$$

$$B_n^1 \longrightarrow B_n^2 \,.$$

Here $H_n^{(1)}$ is the Hankel function, $H_n^{(1)} = J_n + iN_n$.

Without loss of generality we shall limit ourselves to waves traveling in the positive z-direction, that is to β with real part greater than or equal to zero.

It is assumed for the idealized helical wave guide that the electric field tangent to the cylinder surface is continuous at the surface, that the component of the electric field in the helical direction vanishes on the surface and that the component of the magnetic field in the helical direction is continuous

at the surface. The helical direction at the point (a, θ, z) can be designated by the vector

$$\underline{s} = 0\,\underline{v_r} + 1\,\underline{v_\theta} + \delta\,\underline{v_z}$$

where, as before, $\delta = \frac{d}{2\pi a}$, a being the radius of the cylindrical guide and d being the distance between turns on the helix. The boundary conditions can then be expressed as

$$\left.\begin{aligned} \underline{E}(a+)\cdot\underline{s} &= 0 = \underline{E}(a-)\cdot\underline{s} \\ \underline{E}(a+)\cdot\underline{v_z} &= \underline{E}(a-)\cdot\underline{v_z} \\ \underline{H}(a+)\cdot\underline{s} &= \underline{H}(a-)\cdot\underline{s} \end{aligned}\right\} \qquad (2.4)$$

for $r = a$ and all values of θ and z.

Since the equations (2.4) are valid for all values of θ, they are likewise valid for the Fourier coefficients of these quantities taken with respect to θ; that is, they are valid for the corresponding bracketed expressions in Eq.(2.1). The boundary conditions can therefore be written for each n in terms of four linear homogeneous equations in A_n^1, A_n^2, B_n^1, B_n^2.

These equations are

$$\left(\delta - \frac{n\beta}{\gamma^2 a}\right) J_n A_n^1 - \frac{i\mu\omega}{\gamma} J_n' B_n^1 = 0$$

$$\left(\delta - \frac{n\beta}{\gamma^2 a}\right) H_n^{(1)} A_n^2 - \frac{i\mu\omega}{\gamma} H_n^{(1)'} B_n^2 = 0$$

$$J_n A_n^1 - H_n^{(1)} A_n^2 = 0$$

$$\frac{-ik^2}{\mu\omega\gamma} J_n' A_n^1 - \left(\delta - \frac{n\beta}{\gamma^2 a}\right) J_n B_n^1 + \frac{ik^2}{\mu\omega\gamma} H_n^{(1)'} A_n^2 + \left(\delta - \frac{n\beta}{\gamma^2 a}\right) H_n^{(1)} B_n^2 = 0$$

In order that there exist a non-trivial solution the determinant of coefficients must vanish; this leads to the condition

$$\frac{\delta}{\alpha} = \frac{n\beta}{u^2 k} \pm \sqrt{-\frac{J_n'(u)\, H_n^{(1)'}(u)}{u^2\, J_n(u)\, H_n^{(1)}(u)}}\,, \qquad (2.5)$$

where $u = \gamma a$ and $\alpha = ka$. To each solution u of Eq. (2.5) there corresponds a natural mode of the helical wave guide which can be propagated (with or without attenuation) down the guide.

In order that a mode be propagated down the guide without attenuation it is necessary that β be real-valued.

Now by Eq. (2.2)

$$\beta = \sqrt{k^2 - \gamma^2}\ .$$

Hence β will be real-valued if and only if γ is either imaginary-valued or real-valued and less than k.

We now show that <u>there do not exist any real-valued solutions</u> $u = \gamma a$ <u>of Eq. (2.5)</u> less than ak. Since, in this case, both α/β and $n\beta/u^2k$ are real-valued it will be sufficient to show that

$$\frac{J_n'(u)}{J_n(u)} \cdot \frac{H_n^{(1)\prime}(u)}{H_n^{(1)}(u)}$$

is not real-values. Now J_n'/J_n is real-valued. On the other hand

$$\frac{H_n^{(1)\prime}}{H_n} = \frac{J_n' + iN_n'}{J_n + iN_n} = \frac{J_n' J_n + N_n' N_n - i\,(J_n' N_n - N_n' J_n)}{J_n^2 + N_n^2}$$

Since the Wronskian of the Bessel functions, N_n and J_n, is

$$J_n' N_n - N_n' J_n = -\frac{2}{\pi u} \neq 0\ , ^*$$

it follows that $H_n^{(1)\prime}/H_n$ is always complex-valued.

All of the non-attenuated modes of the helical wave guide therefore correspond to imaginary values of γ; the phase velocity of these modes is less than the free-space velocity of the electromagnetic wave. It will be convenient to rewrite Eq. (2.5) in terms of the variable $v = -iu$ and the modified Bessel functions $I_n(v) = i^{-n} J_n(u)$ and $K_n(v) = \frac{\pi}{2}\ i^{n+1} H_n^{(1)}(u)$. Substituting these functions in (2.5) and using $\frac{\beta}{k} = \left[1 - \frac{\gamma^2}{k^2}\right]^{1/2} = \left[1 + \frac{v^2}{\alpha^2}\right]^{1/2}$,

$$\frac{\beta}{\alpha} = \frac{-n}{v^2}\left[1 + \left(\frac{v}{\alpha}\right)^2\right]^{1/2} \pm \sqrt{-\frac{I_n'(v)\,K_n'(v)}{v^2\, I_n(v)\,K_n(v)}}\ . \qquad (2.6)$$

We shall be concerned only with the real solutions of Eq. (2.6)

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* See G. N. Watson: <u>Theory of Bessel Functions</u>, p. 76

The field equations inside the guide ($r \leq a$) for a mode corresponding to a real solution of Eq. (2.6) are found from (2.1) by inserting $I_n(|\gamma r|)$ and taking the n^{th} term. They are

$$
\begin{aligned}
E_r &= \left[\mp \frac{\beta i}{\gamma} C_n I_n'(|\gamma r|) - \frac{nk}{\gamma^2 r} I_n(|\gamma r|) \right] \eta F_n \\
E_\theta &= \left[\pm \frac{n\beta i}{\gamma^2 r} C_n I_n(|\gamma r|) - \frac{k}{\gamma} I_n'(|\gamma r|) \right] \eta F_n \\
E_z &= \mp C_n i I_n(|\gamma r|) \eta F_n \qquad\qquad (2.7) \\
H_r &= \left[\mp \frac{nki}{\gamma^2 r} C_n I_n(\gamma r) + \frac{\beta}{\gamma} I_n'(\gamma r) \right] F_n \\
H_\theta &= \left[\mp \frac{ki}{\gamma} C_n I_n'(|\gamma r|) - \frac{n\beta}{\gamma^2 r} I_n(|\gamma r|) \right] F_n \\
H_z &= I_n(|\gamma r|) F_n \; .
\end{aligned}
$$

where $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ is the impedance of the medium in ohms and $C_n = \sqrt{\frac{-I_n'(|ra|)\,K_n(|ra|)}{I_n(|ra|)\,K_n'(|ra|)}}$

C_n is evaluated by using the fact that only the ratio of the constants in (2.1) can be determined, since the amplitude is arbitrary, and by applying the boundary condition $\underline{E}(a-) \cdot s = 0$ to calculate $\frac{A_n^1}{B_n^1}$. In Eq. (2.7) the upper sign refers to a solution of Eq. (2.6) with a plus sign, whereas the lower sign refers to a minus-sign solution of Eq. (2.6). These field equations correspond to linear combinations of the familiar transverse-electric and transverse-magnetic modes present in ordinary wave guides.

3. <u>The Zero-Order Mode.</u>

In the traveling-wave tube, the zero-order mode has been used to interact with the electron beam. For this reason and because the zero-order mode is somewhat typical of the other modes, the present section will be devoted to a descriptive discussion of the case $n = 0$.

For $n = 0$, Eq. (2.6) becomes

$$\frac{\beta}{\alpha} = \pm \sqrt{\frac{-\,I_0'(v)\,K_0'(v)}{v^2\,I_0(v)\,K_0(v)}} \qquad (3.1)$$

where, as before, $v = -i\gamma a$. We shall study the solutions of this equation with the help of the function

$$w_o(v) = \sqrt{\frac{-I_o'(v)\ K_o'(v)}{v^2\ I_o(v)\ K_o(v)}} \tag{3.2}$$

It is clear that the set of values of v for which $w_o(v)$ is equal to either $\frac{\delta}{\alpha}$ or $-\frac{\delta}{\alpha}$ is a solution of Eq. (3.1).

As we shall see, $w_o(v)$ is a positive monotonically decreasing function of v for $v \geq 0$. As v becomes infinite, the graph of $w_o(v)$ approaches the v-axis from above. Figure 2 is a graph of the function w_o. The function has a logarithmic singularity at the origin; it is regular at all points along the positive v-axis. The asymptotic expansion for w_o is of the form

$$w_o(v) = \frac{1}{v} + \frac{1}{4}\frac{1}{v^3} + \ldots . \tag{3.3}$$

This expansion holds very well even for values of v of the order of 2 or 3. It is apparent from the asymptotic expansion that for small $\frac{\delta}{\alpha}$, say less than 0.2, the solution to Eq. (3.1) can be approximated as in Eq. (1.3).

The field equations inside of the guide or the zero-order mode are simply

$$E_r = \frac{-ki}{\gamma} C_o I_o'(|\gamma r|)\, \eta\, F_o , \qquad H_r = \frac{\beta}{\gamma} I_o'(|\gamma r|)\, F_o ,$$

$$E_\theta = -\frac{k}{\gamma} I_o'(|\gamma r|)\, F_o , \qquad H_\theta = -\frac{ki}{\gamma} C_o I_o'(|\gamma r|)\, F_o ,$$

$$E_z = -i\, C_o I_o(|\gamma r|)\, \eta\, F_o ; \qquad H_z = I_o(|\gamma r|)\, F_o .$$

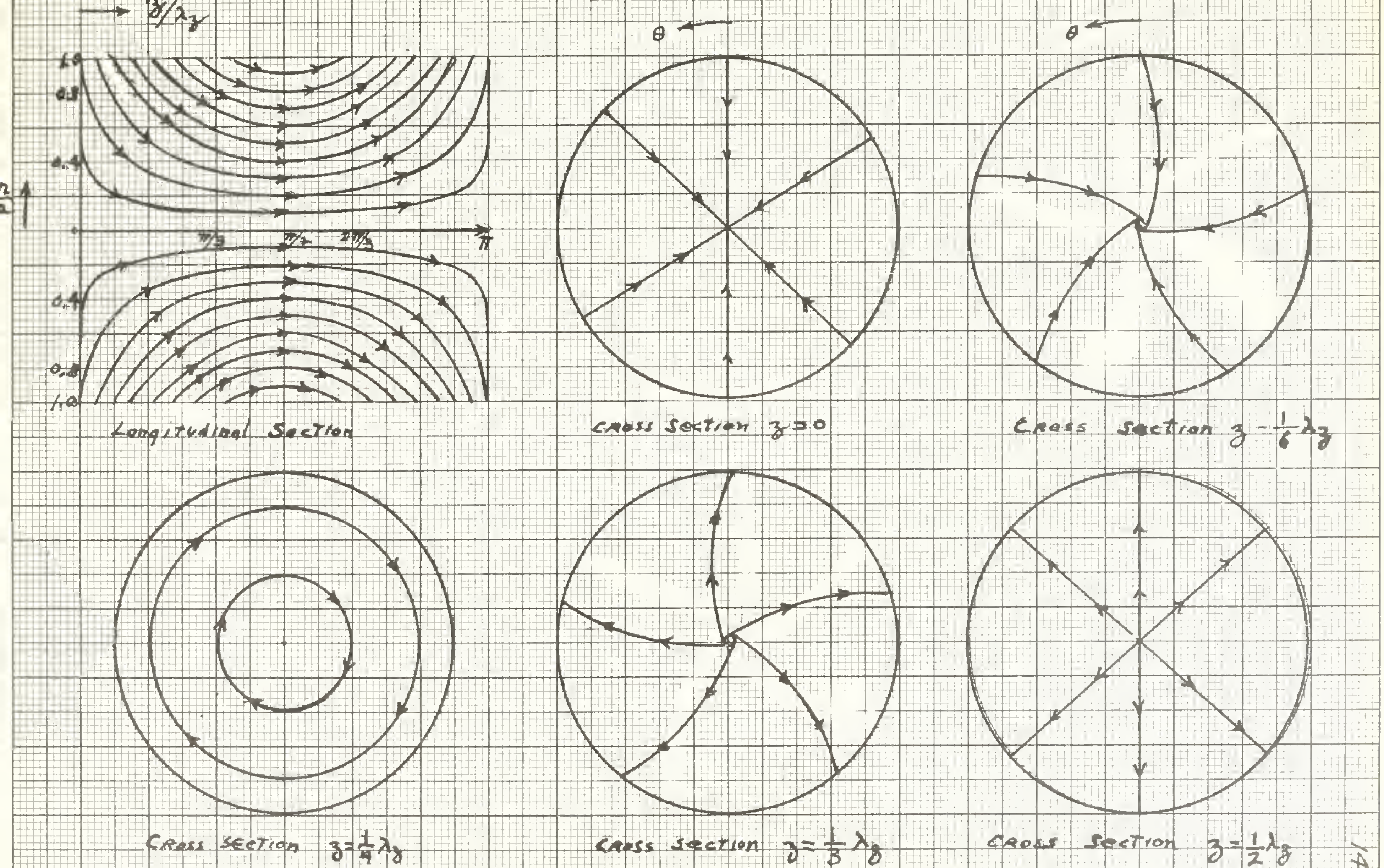
where $C_o = \frac{I_1(|\gamma a|)\ K_o(|\gamma a|)}{I_o(|\gamma a|)\ K_1(|\gamma a|)}$. The factor i can be interpreted as a phase advance of a quarter of a wave-lenght (i.e. $\lambda_z/4$) in the z-direction. In practice C_o is close to one. When this is the case, the E and H fields differ only by the factor η and the fact that E lags H by a quarter of a wave-length in the z-direction.

Sectional drawings of a zero-order E-field are shown in Fig. 3; a graph of E_z as a function of r is given in Fig. 4. The parameters for the helical wave guide illustrated in these figures are

$$\delta = \frac{1}{16} \quad ; \quad \alpha = \frac{1}{4}$$

ZERO ORDER MODE
SECTION DRAWINGS OF E FIELD

FIG. 3



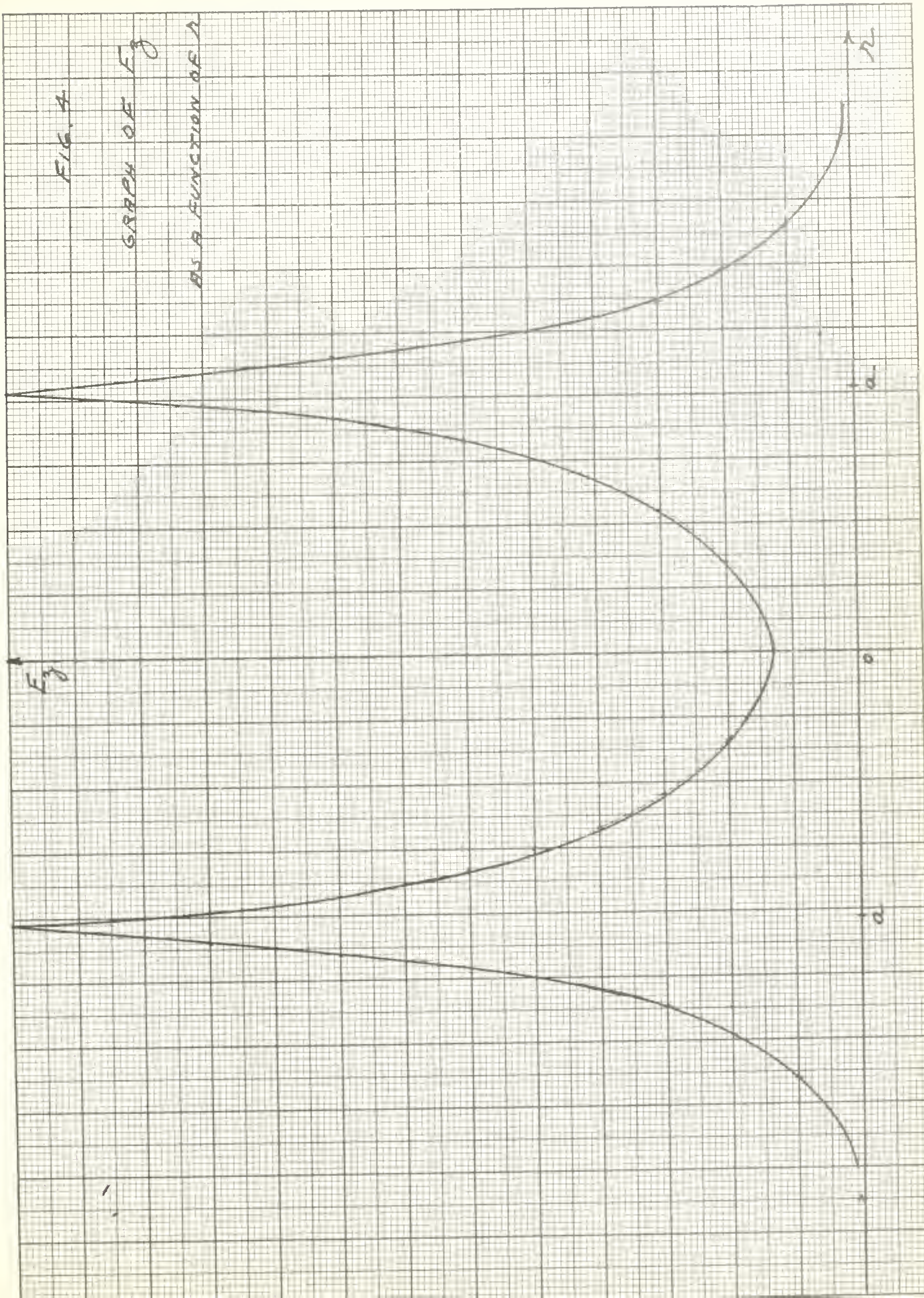


FIG. 4 GRAPH OF E_y AS A FUNCTION OF R

The helical windings form a left-hand screw thread for which $d/a = \frac{2\pi}{16} = 0.39$. As can easily be computed from Fig. 2,

$$a\gamma = 3.95\ i$$

Hence

$$\beta/k = \frac{\lambda_o}{\lambda_z} = 15.8 \qquad \text{and} \qquad C_o = 0.875 \quad .$$

Fig. 3 shows the E-field for only half of a wave-length since the remaining half is the same except for sign.

4. Mathematical Introduction to the Discussion of Higher Modes.

We are interested only in the non-attenuated electromagnetic modes propagated along a helical wave guide. As we have seen in Sec. 2, these modes correspond to the real solutions of the equation

$$\frac{\gamma}{\alpha} = -\frac{n}{v^2}\frac{\beta}{k} \pm \sqrt{-\frac{I_n'(v)\,K_n'(v)}{v^2\,I_n(v)\,K_n(v)}} \qquad (n = 0, \pm 1, \pm 2,.) \quad (2.6)$$

where β is the positive square root of $(k^2 - \gamma^2)$. We shall study the solutions of these equations by investigating the functions on the right-hand side for $v \geq 0$.

For the modified Bessel functions

$$I_{-n}(v) = c_{1,n}\, I_n(v) \qquad \text{and} \qquad K_{-n}(v) = c_{2,n}\, K_n(v)$$

where $c_{1,n}$ and $c_{2,n}$ are constants. We can therefore replace $\frac{I_{-n}'(v)\,K_{-n}'(v)}{I_{-n}(v)\,K_{-n}(v)}$ by $\frac{I_n'(v)\,K_n'(v)}{I_n(v)\,K_n(v)}$. This permits us to consider the solutions of Eq. (2.6) for positive and negative values of n simultaneously by finding the real solutions of

$$\pm\frac{\gamma}{\alpha} = -\frac{n\beta}{v^2 k} \pm \sqrt{-\frac{I_n'(v)\,K_n'(v)}{v^2\,I_n(v)\,K_n(v)}} \qquad (n = 0,1,2,\ldots.) \quad . \quad (4.1)$$

If the plus sign holds on the left for a solution, then the solution corresponds to an n-mode, whereas if the minus sign holds on the left, it corresponds to a (-n)-mode. Hence aside from sign, we need only study the two sets of functions

$$W_{1,n}(v) = \sqrt{-\frac{I_n'(v)\,K_n'(v)}{v^2\,I_n(v)\,K_n(v)}} + \frac{n}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}}$$

$$W_{2,n}(v) = \sqrt{-\frac{I_n'(v)\,K_n'(v)}{v^2\,I_n(v)\,K_n(v)}} - \frac{n}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}} \qquad (4.2)$$

for $n = 0,1,2$ and $v > 0$.

We are interested in $W_{j,n}(v)$ $(j = 1,2)$ as a function of both n and v. As a function of n, we shall show that the function $W_{1,n}$ is monotonic increasing, whereas the function $W_{2,n}$ is monotonic decreasing. This tells us that the modes are separated, and, when the functions are monotonic functions of v, that the solutions of Eq. (4.1) are themselves ordered.

We have also tried to discover the behavior of the functions as v varies. In this we have been only partially successful. It is easy to show that $W_{1,n}(v)$ is a positive monotonic decreasing function of v approaching the v-axis asymptotically as $v \to \infty$. Our most important result for the other set of functions is that for $\alpha < \sqrt[4]{n(n-1)^2 (n-2)}$, $W_{2,n}(v)$ is a negative monotonic increasing function of v approaching the v-axis asymptotically as $v \to \infty$. When the function is monotonic, there exists either no solution or exactly one solution for Eq. (4.1) depending simply on the initial value of the function for $v = 0$.

The principal difficulty in handling the functions $W_{j,n}(v)$ $(j = 1,2)$ stems from the fact that the Bessel functions appear under a radical sign. Now $\sqrt{-\dfrac{I_n'(v)\, K_n'(v)}{v^2\, I_n(v)\, K_n(v)}}$ is the geometric mean of the functions $\dfrac{I_n'(v)}{vI_n(v)}$ and $-\dfrac{K_n'(v)}{vK_n(v)}$; this suggests that we work with the functions

$$x_n(v) = \frac{1}{v} \frac{I_n'(v)}{I_n(v)} \qquad \text{and} \qquad y_n(v) = \frac{-1}{v} \frac{K_n'(v)}{K_n(v)} \tag{4.3}$$

instead of

$$w_n(v) = \sqrt{-\frac{I_n'(v)\, K_n'(v)}{v^2\, I_n(v)\, K_n(v)}} \quad . \tag{4.4}$$

There is, however, a very good physical reason for adopting this approach. If we choose slightly different boundary conditions for an idealized helical wave guide, namely, conditions which isolate the fields inside and outside the guide* , then we obtain instead of Eq. (2.6) the equation

$$\frac{\delta}{\alpha} = -\frac{n}{v^2} \frac{\beta}{k} \pm \frac{I_n'(v)}{vI_n(v)} \tag{4.5a}$$

for the modes inside of the guide and

$$\frac{\delta}{\alpha} = -\frac{n}{v^2} \frac{\beta}{k} \pm \frac{K_n'(v)}{vK_n(v)} \tag{4.5b}$$

*On the cylindrical guide both E and H are taken to be zero in the helical direction.

for the modes outside of the guide. It is intuitively clear on physical grounds that the phase velocities and hence the solutions of Eq. (2.6),(4.5a) and (4.5b)) for corresponding modes in the three different types of helical guides should be close together.

In all frankness, however, it should be stated that for the authors of this paper the main suggestion for attempting this approach came from the fact that in an earlier paper by one of them* the field inside the modified helical guide had been successfully dealt with by studying equation (4.5a). The results of this earlier paper essentially comprise sections 5 and 6.

The first part of our development will therefore not be concerned with the functions $W_{1,n}$ and $W_{2,n}$, but rather with the functions associated with Eqs. (4.5a) and (4.5b):

$$\begin{aligned} X_{1,n}(v) &= x_n(v) + \frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} \quad , \\ X_{2,n}(v) &= x_n(v) - \frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} \quad , \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} Y_{1,n}(v) &= y_n(v) + \frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} \quad , \\ Y_{2,n}(v) &= y_n(v) - \frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} \quad . \end{aligned} \tag{4.7}$$

As we shall see, both the methods and the machinery developed for the X's and Y's will be applicable to the W's.

Section 5 is devoted to a study of the functions $X_{1,n}$ and $Y_{1,n}$. It is shown that

$$x_{n+1} > x_n \qquad \text{and} \qquad y_{n+1} > y_n \quad .$$

We obtain as an immediate consequence of this not only

$$X_{1,n+1} > X_{1,n} \qquad \text{and} \qquad Y_{1,n+1} > Y_{1,n}$$

but also

$$W_{1,n+1} > W_{1,n} \quad .$$

* R. S. Phillips, A Helical Wave Guide, New York University, Washington Square College Mathematical Research Group, special report No. 170-2.

It is also shown in Sec. 5, that x_n and y_n (and hence w_n) are positive monotonic decreasing functions. From this it follows that $X_{1,n}$, $Y_{1,n}$, and $W_{1,n}$ are positive monotonic decreasing functions of v.

The functions $X_{2,n}$ and $Y_{2,n}$ are treated in sections 6 and 7 respectively. It is first shown that

$$x_{n+1} - x_n < 1/v^2 \quad \text{and} \quad y_{n+1} - y_n < 1/v^2 \quad . \tag{4.8}$$

One obtains as a consequence of (4.8) that

$$X_{2,n+1} < X_{2,n} \quad \text{and} \quad Y_{2,n+1} < Y_{2,n} \quad .$$

In order to prove that $W_{2,n+1} < W_{2,n}$, we show in Sec. 9 that

$$(w_{n+1} - w_n) < 1/v^2 \ . \tag{4.9}$$

In this case, however, the inequalities of (4.8) are not sufficient to give (4.9); it is necessary to derive a stronger result, namely,

$$(x_{n+1} - x_n) \leq \frac{1}{v^2} \frac{1}{\sqrt{1 + \frac{v^2}{(n+1)(n+2)}}} \quad \text{and} \quad (y_{n+1} - y_n) < \frac{1}{v^2} \frac{1}{\sqrt{1 + \frac{v^2}{(n + 1/2)^2}}}$$

These inequalities are obtained in Sec. 8.

By far the most difficult theorems in the present paper are concerned with the behavior of $X_{2,n}$ and $Y_{2,n}$ as functions of v. It is shown that for $\alpha \leq n$, $X_{2,n}$ is negative and monotonic increasing; for $n < \alpha < \sqrt[4]{n(n+1)^2 (n+2)}$, $X_{2,n}$ has a single maximum; and for $\sqrt[4]{n(n+1)^2 (n+2)} \leq \alpha$, $X_{2,n}$ is positive and monotonic decreasing. Likewise for $\alpha \leq \sqrt[4]{n(n-1)^2 (n-2)}$, $Y_{2,n}$ is negative and monotonic increasing; for $\sqrt[4]{n(n-1)^2 (n-2)} < \alpha < n$, $Y_{2,n}$ has a single minimum; and for $n \leq \alpha$, $Y_{2,n}$ is positive and monotonic decreasing. These results are summarized in Figs. 6 and 8.

Let us define

$$\varphi_n(v,\alpha) = \frac{n}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}} \quad . \tag{4.10}$$

Then as an immediate consequence of the above theorems, we obtain the following inequalities:

$$\begin{aligned}
&\varphi_n(v,\sqrt{n(n+1)}\,) < x_n(v) < \varphi_n(v,n) , \\
&\varphi_n(v,n) < y_n(v) < \varphi_n(v, \sqrt{n(n-1)}\,) , \\
&\varphi_n'(v,n) < x_n'(v) < \varphi_n'(v, \sqrt[4]{n(n+1)^2(n+2)}\,) \\
&\varphi_n'(v, \sqrt[4]{n(n-1)^2(n-2)}\,) < y_n'(v) < \varphi_n'(v,n) .
\end{aligned} \tag{4.11}$$

for $v > 0$. We have made use of these inequalities in Sec. 9 in order to study the behavior of $W_{2,n}$ as a function of v. In this connection the strongest result which we were able to obtain was that $W_{2,n}(v)$ is negative and monotonic increasing for $\alpha \leq \sqrt[4]{n(n-1)^2(n-2)}$, and positive and monotonic decreasing for $\alpha \geq \sqrt[4]{n(n+1)^2(n+2)}$. We have attempted to fill in the gap by means of graphs for the cases n = 1,2, and 3.

Practically all of our results have been obtained from a study of the differential equations satisfied by x_n, y_n, and associated functions. By making use of the defining differential equation for the Bessel functions, one readily finds that these differential equations are of the Riccati type. For most of our arguments it is sufficient to determine the vector field induced by the differential equation along the v-axis. For example, a function which is positive at v = 0 can become negative only by crossing the v-axis with a non-positive slope; if the differential equation satisfied by the function has a positive slope for all $v > 0$, then we may conclude that the function is positive for all $v \geq 0$. Only the theorems concerned with a characterization of $X_{2,n}$ and $Y_{2,n}$ as functions of v require a more detailed investigation of the vector field.

5. The Functions $X_{1,n}$ and $Y_{1,n}$.

As in section 4, we let

$$x_n = \frac{1}{v}\,\frac{I_n'(v)}{I_n(v)} , \quad y_n = -\,\frac{1}{v}\,\frac{K_n'(v)}{K_n(v)} \tag{4.3}$$

and

$$X_{1,n}(v) = \frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} + x_n(v) , \tag{4.6}$$

$$Y_{1,n}(v) = \frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} + y_n(v) . \tag{4.7}$$

It will be shown in the present section that $X_{1,n}$ and $Y_{1,n}$ $(n \geq 1)$ are of the order of $\frac{1}{v^2}$ for v near zero, are always positive and monotonic decreasing, and their graphs approach the real axis asymptotically as v approaches infinity. Moreover, it will be established that each of the sequences of functions $X_{1,n}$ and $Y_{1,n}$ is an increasing function of n.

In order to demonstrate the theorems mentioned above, it is necessary to prove similar results concerning the functions x_n and y_n.

If use is made of the fact that $I_n(v)$, $K_n(v)$ satisfy Bessel's differential equation, it is readily seen that the functions x_n, y_n satisfy the following Riccati differential equations:

$$\frac{dx_n}{dv} = - x_n^2\, v - \frac{2}{v}\, x_n + \frac{1}{v} + \frac{n^2}{v^3}\,, \tag{5.1}$$

$$\frac{dy_n}{dv} = y_n^2\, v - \frac{2}{v}\, y_n - \frac{1}{v} - \frac{n^2}{v^3} \tag{5.2}$$

The expansions for x_n, y_n near the origin are:

$$x_n = \frac{n}{v^2}\left[1 + \frac{2}{n(n+1)}\left(\frac{v}{2}\right)^2 - \frac{2}{n(n+1)^2(n+2)}\left(\frac{v}{2}\right)^4 + \ldots\right] \tag{5.3}$$

$$y_n = \frac{n}{v^2}\left[1 + \frac{2}{n(n-1)}\left(\frac{v}{2}\right)^2 - \frac{2}{n(n-1)^2(n-2)}\left(\frac{v}{2}\right)^4 + \ldots\right] \tag{5.4}$$

$$y_1 = \frac{1}{v^2} - \log\frac{\gamma v}{2} + \ldots \tag{5.4a}$$

$$y_0 = -\frac{1}{v^2\log\frac{\gamma v}{2}} + \ldots \tag{5.4b}$$

where $\log\gamma = .577$.

The asymptotic expansions for x_n, y_n are:

$$x_n = \frac{1}{v} - \frac{1}{2v^2} + \frac{1}{2}\left(n^2 - \frac{1}{4}\right)\frac{1}{v^3} + \frac{1}{2}\left(n^2 - 4\right)\frac{1}{v^4} + \ldots\,, \tag{5.5}$$

$$y_n = \frac{1}{v} + \frac{1}{2v^2} + \frac{1}{2}\left(n^2 - \frac{1}{4}\right)\frac{1}{v^3} - \frac{1}{2}\left(n^2 - 4\right)\frac{1}{v^4} + \ldots \tag{5.6}$$

These expansions with the exception of (5.4a) and (5.4b) can be obtained directly from (5.1) and (5.2). The expansions (5.4a) and (5.4b) are found with the aid of corresponding expansions for K_0, K_1 and K_2.

Eqs. (5.1)-(5.6) enable us to establish a series of lemmas from which our principal theorems will follow.

Lemma 1. x_n is a positive monotonic decreasing function of v for all $v > 0$.

By Eq. (5.3), x_n is positive for sufficiently small v. Moreover, from known properties of Bessel functions, x_n is an analytic function for $v > 0$. Consequently, if x_n ever becomes negative, there exists a smallest v, say v_1, such that $x_n(v_1) = 0$. By Eq. (5.1) we would have

$$\left(\frac{dx_n}{dv}\right)_{v = v_1} = \frac{1}{v_1} + \frac{n^2}{v_1^3} > 0 .$$

That is to say, x_n is an increasing function of v at v_1. But this is contrary to the fact that x_n is positive for small v. Therefore x_n is positive for all $v \geq 0$.

The second part of the lemma is proved in the following way. Equation (5.3) shows that $\frac{dx_n}{dv}$ is negative for small $v > 0$. Furthermore the derivative of Eq. (5.1) is

$$\frac{d^2 x_n}{dv^2} = -\left(2 x_n v + \frac{2}{v}\right) \frac{dx_n}{dv} - x_n^2 + \frac{2}{v^2} x_n - \frac{1}{v^2} - \frac{3n^2}{v^4} . \tag{5.7}$$

If $\frac{dx_n}{dv}$ were ever to vanish, say at v_1, then Eq. (5.1) combines with Eq. (5.7) to give

$$\left(\frac{d^2 x_n}{dv^2}\right)_{v = v_1} = -2 x_n^2(v_1) - \frac{2n^2}{v_1^4} < 0.$$

Hence, if $\frac{dx_n}{dv}$ were to vanish at v_1, it would have to change from positive to negative values. Again this is contrary to the fact that $\frac{dx_n}{dv}$ is initially negative. Hence $\frac{dx_n}{dv}$ must be negative for all values of $v > 0$.

<u>Lemma 2.</u> $x_n < x_{n+1}$ <u>for all</u> $n \geq 0$ <u>and</u> $v \geq 0$.

Let $\Delta x = x_{n+1} - x_n$. By Eq. (5.1)

$$\frac{d(\Delta x)}{dv} = - v\ (x_{n+1} + x_n)\ \Delta x - \frac{2}{v}\ \Delta x + \frac{2n+1}{v^3} \qquad . \qquad (5.8)$$

By Eq. (5.3), Δx behaves like $\frac{1}{v^2}$ for small v. Furthermore, it cannot become 0, since if Δx should vanish, say at $v = v_1$, then Eq. ((5.b) becomes

$$\frac{d(\Delta x)}{dv} = \frac{2n+1}{v^3} > 0 \quad \text{for } v = v_1 \quad ,$$

which is impossible since Δx is initially positive.

<u>Lemma 3.</u> y_n <u>is a positive monotonic decreasing function of</u> v <u>for all</u> $v > 0$.

By Eq. (5.4), y_n is positive for small v, it is an analytic function of v for $v > 0$. If y_n were to become negative, there would exist a $v = v_1$ such that $y_n(v_1) = 0$. Equation (5.2) for this value of v becomes

$$\left(\frac{dy_n}{dv}\right)_{v=v_1} = - \frac{1}{v_1} - \frac{n^2}{v_1^3} < 0. \qquad (5.9)$$

Hence, although y_n could presumably become 0 and take on negative values, (5.9) shows that if it did so, it could never become positive again. By the asymptotic formula (5.6) however, y_n approaches the v-axis from above as v approaches infinity. This behavior is consistent only with the assumption that $y_n > 0$ for all $v > 0$.

To go on to the second part of the lemma, we see from Eq. (5.4) that $\frac{dy_n}{dv}$ is negative for v small. Differentiating Eq. (5.2) we get

$$\frac{d^2y_n}{dv^2} = (2vy_n - \frac{2}{v})\ \frac{dy_n}{dv} + y_n{}^2 + \frac{2}{v^2}\ y_n + \frac{1}{v^2} + \frac{3n^2}{v^4} \ . \qquad (5.10)$$

For any $v = v_1$ for which $\frac{dy_n}{dv} = 0$, equations (5.10) and (5.2) combine to give

$$\left(\frac{d^2y_n}{dv^2}\right)_{v=v_1} = 2\ y_n{}^2\ (v_1) + \frac{2n^2}{v_1^4} > 0 \quad .$$

Hence, if $\frac{dy_n}{dv}$ were ever to vanish, it must change from negative to positive values. That is to say if $\frac{dy_n}{dv}$ ever becomes positive, it must remain so. This is inconsistent with the asymptotic behavior of y_n as v approaches infinity, as seen from (5.6). Hence $\frac{dy_n}{dv}$ is negative for all $v > 0$.

<u>Lemma 4</u>. $y_n < y_{n+1}$ <u>for all</u> $n \geq 0$ <u>and all</u> $v \geq 0$.

Let $\Delta y = y_{n+1} - y_n$. Equation (5.2) yields the differential equation for Δy, namely,

$$\frac{d(\Delta y)}{dv} = v(y_{n+1} + y_n)\Delta y - \frac{2}{v}\Delta y - \frac{2n+1}{v^3} \qquad (5.11)$$

If Δy were ever to vanish, say for $v = v_1$, then Eq. (5.11) reduces to

$$\left(\frac{d(\Delta y)}{dv}\right)_{v=v_1} = -\frac{2n+1}{v_1^3} < 0 . \qquad (5.12)$$

Again Δy behaves like $\frac{1}{v^2}$ for small $v > 0$. If Δy were ever zero, then, by Eq. (5.12), it would thereafter have to remain negative. This, however, is impossible since (5.6) yields

$$\Delta y = \frac{2n+1}{2}\frac{1}{v^3} + \cdot > 0 \quad \text{for large } v.$$

Hence Δy is always positive.

Lemmas 1 - 4 allow us to prove the following theorems.

<u>Theorem 1</u>. $X_{1,n} = \frac{n}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}} + x_n$ <u>is a positive monotonic decreasing function of</u> v <u>for all</u> $v > 0$ <u>and all</u> $n \geq 0$.

$X_{1,n}$ is clearly the sum of two positive monotonic decreasing functions of v.

In the same way, we have

<u>Theorem 2</u>. $Y_{1,n} = \frac{n}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}} + y_n$ <u>is a positive monotonic decreasing function of</u> v <u>for all</u> $v > 0$ <u>and all</u> $n \geq 0$.

If now ΔX be defined by the equation

$$\Delta X = X_{1,n+1} - X_{1,n} ,$$

then,

$$\Delta X = \frac{1}{v^2}\left[1 + \frac{v^2}{\alpha^2}\right]^{1/2} + \Delta x \ .$$

ΔX is therefore seen to be the sum of two positive functions, which proves

<u>Theorem 3.</u> $X_{1,n} < X_{1,n+1}$ <u>for</u> <u>all</u> $n \geq 0$ <u>and</u> $v \geq 0$.

In the same manner, we can show that

$$\Delta Y = Y_{1,n+1} - Y_{1,n} = \frac{1}{v^2}\left[1 + \frac{v^2}{\alpha^2}\right]^{-1/2} + \Delta y$$

is positive. Thus we have

<u>Theorem 4.</u> $Y_{1,n} < Y_{1,n+1}$ <u>for</u> <u>all</u> $n \geq 0$ <u>and</u> $v \geq 0$.

A summary of the results of section 5 is shown in graphical form in Fig. 5.

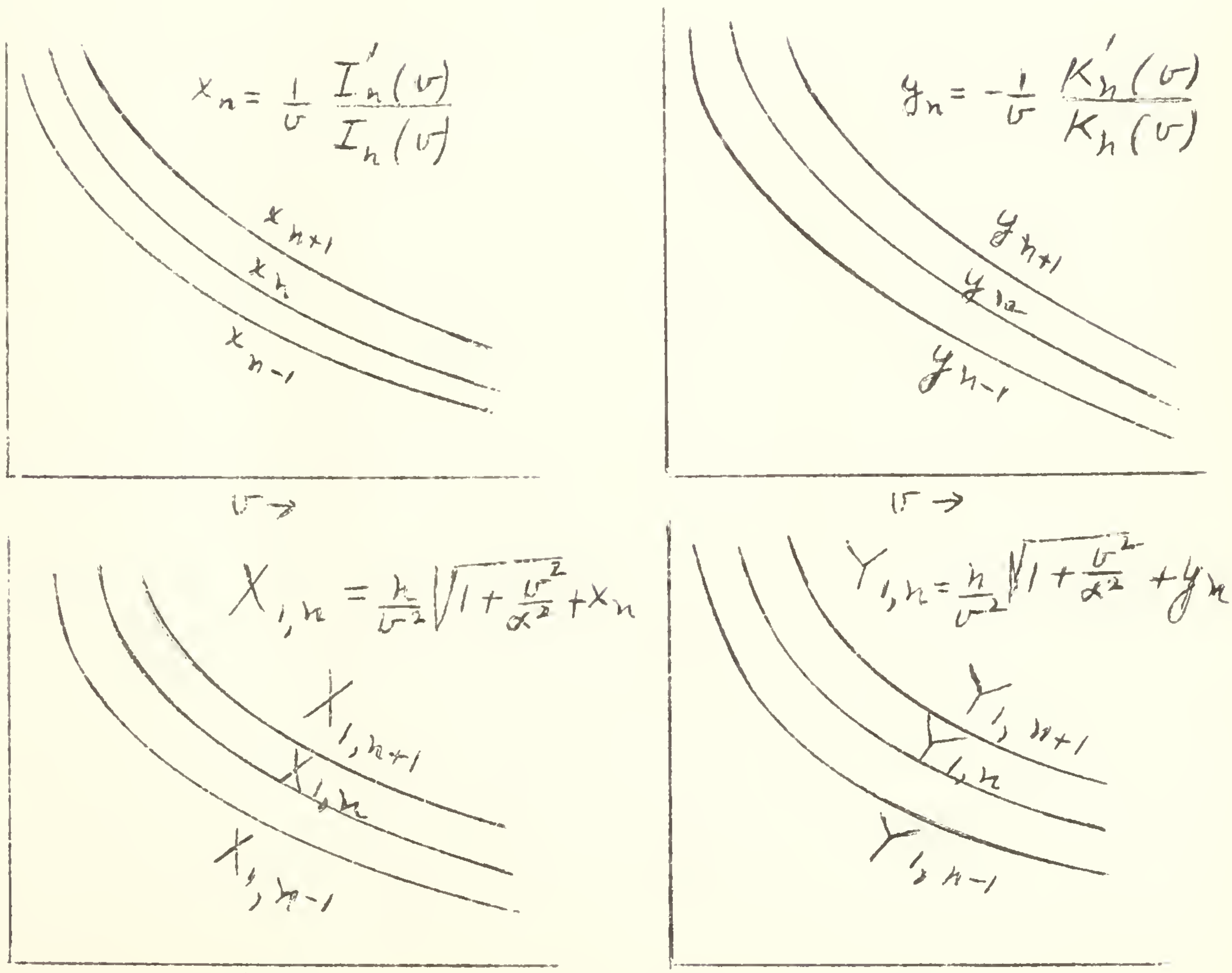


FIG 5

6. The Function $X_{2,n}$.

The present section is devoted to a study of the functions $X_{2,n}(v)$:

$$X_{2,n}(v) = X_{2,n} = - \frac{n}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}} + x_n \qquad (4.6)$$

We shall show that the functions $X_{2,n}$ form an ordered family with respect to n. The function $X_{2,n}$ is the difference of two monotonic functions. It can therefore be expected that its behavior is much more complicated than that of $X_{1,n}$. It is possible, however, to prove a theorem characterizing $X_{2,n}(v)$ for all values of the two parameters n and α. As a corollary of this theorem we obtain upper and lower bounds on x_n and its derivative.

Let

$$\bar{X}_n = x_n - \frac{n}{v^2} \qquad (6.1)$$

One readily obtains from Eqs. (5.1), (5.3), (5.5) the corresponding equations for $\bar{X}_n$, viz.,

$$\frac{d\bar{X}_n}{dv} = - \bar{X}_n^2\, v - \frac{2}{v}(n+1)\ \bar{X}_n + \frac{1}{v}\ , \qquad (6.2)$$

$$\bar{X}_n = \frac{1}{2(n+1)} - \frac{v^2}{8(n+1)^2(n+2)} + \dots \text{ for } n \geq 0 \text{ and } v \text{ small,} \qquad (6.3)$$

and

$$\bar{X}_n = \frac{1}{v} - \frac{1}{2}\,\frac{(2n+1)}{v^2} + \cdots \text{ for } n \geq 0 \text{ and } v \text{ large.} \qquad (6.4)$$

The proof of the ordering theorem is an easy consequence of the two following lemmas:

Lemma 5. $\bar{X}_n$ is a positive monotonic decreasing function of v for all $v > 0$.

Lemma 6. $\bar{X}_n > \bar{X}_{n+1}$, for all $n \geq 0$, $v \geq 0$.

The proof of lemma 5 is very similar to that of lemma 1. It is clear, from Eq. (6.3) that $\bar{X}_n$ is positive for small values of v. Further, it can never vanish for a positive v, since if it were zero, say at v_1, then by Eq. (6.2),

$\left(\frac{d\bar{X}_n}{dv}\right)_{v_1} = \frac{1}{v_1} > 0$. Similarly, $\frac{d\bar{X}_n}{dv}$ is initially negative and it, too, cannot become zero since, if it did one can easily show that $\frac{d^2\bar{X}_n}{dv^2} = -2\bar{X}_n^2 < 0$ at such a point.

The proof of lemma 6 proceeds as follows:

Let $\Delta\bar{X} = \bar{X}_{n+1} - \bar{X}_n$. It follows from Eq. (6.3) that $\Delta\bar{X} = \frac{-1}{2(n+1)(n+2)}$ for $v = 0$. Hence, $\Delta\bar{X}$ is negative for small v. The differential equation for $\Delta\bar{X}$ can easily be derived from Eq. (6.2).
It is

$$\frac{d(\Delta\bar{X})}{dv} = -v\,(\bar{X}_n + \bar{X}_{n+1})\,\Delta\bar{X} - \frac{2(n+2)}{v}\,\Delta\bar{X} - \frac{2}{v}\,\bar{X}_n \quad .$$

Again $\Delta\bar{X}$ can not become zero, for if it were ever to vanish, say for $v = v_1$, then for this value of v, $\frac{d(\Delta\bar{X})}{dv} = -\frac{2}{v}\bar{X}_n < 0$ by lemma 5.

Theorem 5. $X_{2,n} > X_{2,n+1}$ for all $n \geq 0$ and $v \geq 0$.

Proof: Let $\Delta X = X_{2,n+1} - X_{2,n}$, then from Eq. (4.6) and the definition of $\Delta\bar{X}$, we can write ΔX as follows:

$$\Delta X = \frac{1}{v^2}\left\{1 - \left[1 + \frac{v^2}{\alpha^2}\right]^{1/2}\right\} + \Delta\bar{X} \quad .$$

Hence ΔX is the sum of two negative functions which proves the theorem.

The functions $X_{2,n}$ are not necessarily of one sign nor monotonic. Nevertheless, as we shall show, these functions are either monotonic increasing, monotonic decreasing, or have a single maximum, depending on the values of n and .

The series expansion and the asymptotic expansion of $X_{2,n}$ obtained from the definition of $X_{2,n}$ and Eq. (5.3) and (5.5), are:

$$X_{2,n}(v) = \left(\frac{1}{2(n+1)} - \frac{n}{2\alpha^2}\right) - \left(\frac{1}{8(n+1)^2(n+2)} - \frac{n}{8\alpha^4}\right)v^2 + \ldots \qquad (6.5)$$

$$X_{2,n}(v) = \left(1 - \frac{n}{\alpha}\right)\frac{1}{v} - \frac{1}{2}\left(\frac{1}{v}\right)^2 + \ldots \qquad (6.6)$$

As seen from Eq. (5.3), the poles of $\frac{n}{v^2}\sqrt{1 + (\frac{v}{\alpha})^2}$ and x_n, cancel each other leaving $X_{2,n}$ regular at the origin. From Eq. (6.6) it is seen that as $v \to \infty$, the graph of $X_{2,n}$ approaches the v-axis from below if $\alpha \leq n$ and from above if $\alpha > n$. Furthermore, from (6.5), the slope for sufficiently small $v > 0$ can be seen to be positive for $\alpha < \sqrt[4]{n(n+1)^2(n+2)}$ and negative otherwise. These remarks are consistent with the following theorem:

Theorem 6. If $\alpha \leq n$, then $X_{2,n}$ is negative and monotonic increasing, and its graph approaches the v-axis from below; if $n < \alpha < \sqrt[4]{n(n+1)^2(n+2)}$, then $X_{2,n}$ has exactly one maximum and no minima, and its graph thereafter approaches the the v-axis from above; if $\alpha \geq \sqrt[4]{n(n+1)^2(n+2)}$, then $X_{2,n}$ is positive and monotonic decreasing, and its graph approaches the v-axis from above.

This theorem is valid for all $n \geq 0$ and $v > 0$. The theorem is illustrated by the graph in Fig. 6.

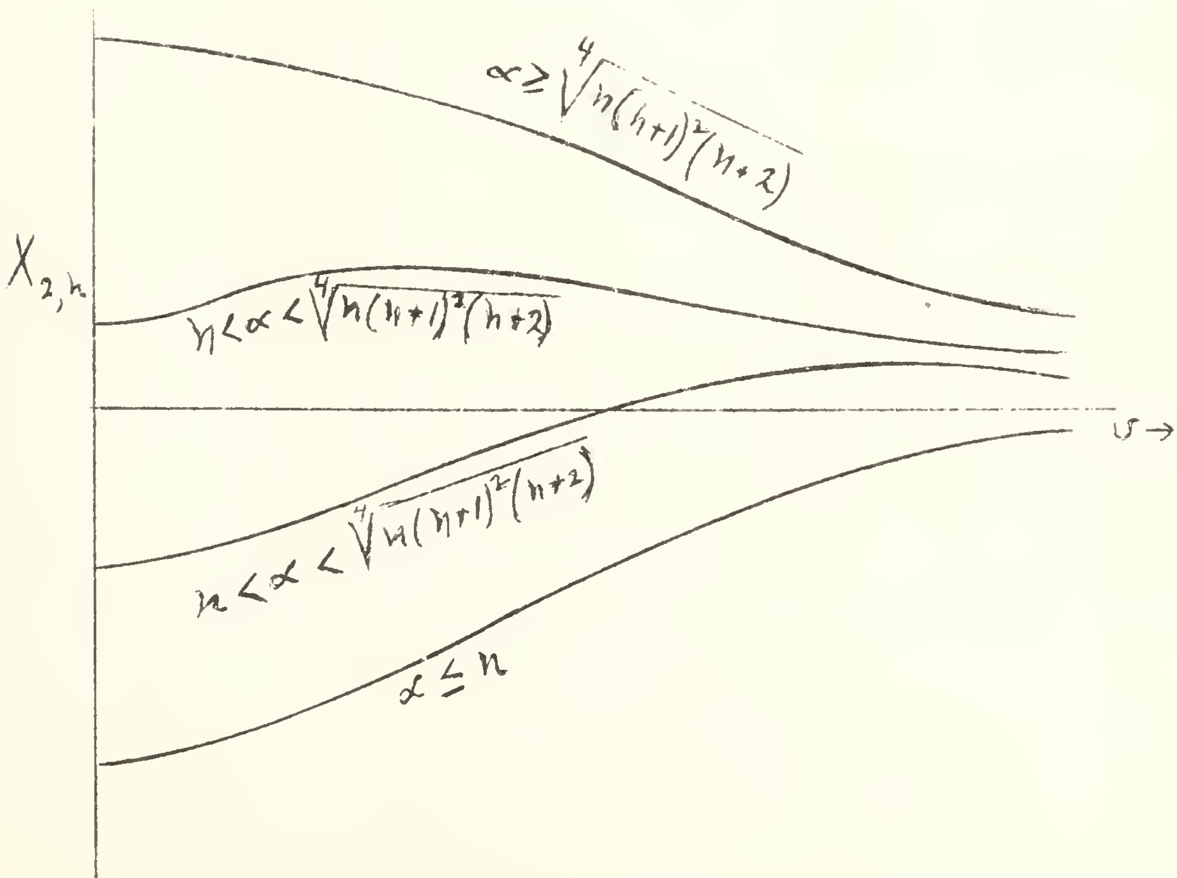


FIG. 6

If we set $x_n = \frac{n}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}} + Z$ and substitute in Eq. (5.1), we obtain the differential equation

$$\frac{dZ}{dv} = -vZ^2 - \frac{2}{v}\left\{n\left[1 + \left(\frac{v}{\alpha}\right)^2\right]^{1/2} + 1\right\} Z + \frac{1}{v}$$

$$- \frac{n^2}{v} \cdot \frac{1}{\alpha^2} - \frac{n}{v} \cdot \frac{1}{\alpha^2} \quad \frac{1}{\left[1 + \left(\frac{v}{\alpha}\right)^2\right]^{1/2}} \quad , \qquad (6.7)$$

which is clearly satisfied by the function $X_{2,n}$. The differential equation (6.7) associates a direction to every point in the (v,Z)-plane. The point and associated direction are known as a line-element. In particular we are interested in the locus of all points where the slope of the line element is zero. Setting $\frac{dZ}{dv}$ equal to zero results in a quadratic equation in Z. The roots of this quadratic in Z are two functions of v. Let $U(v)$ be the greater and $L(v)$ be the smaller of these two roots. The curves representing these two functions divide the half plane $v \geq 0$ into three regions: the region above the U-curve in which $\frac{dZ}{dv}$ is negative; the region between the U and L - curves in which $\frac{dZ}{dv}$ is positive; and the region below the L-curve in which $\frac{dZ}{dv}$ is again negative.

Most of the proof is concerned with an investigation of the U-and L - curves. The behavior of the $X_{2,n}$ curve for small v is known by Eq. (6.5). We know therefore to which of the three regions $X_{2,n}$ belongs initially. Recall that $X_{2,n}$ belongs to the family of Z-curves. It follows that $X_{2,n}$ will be monotonic increasing (decreasing) until it intersects with zero slope, a U- or L-curve, after which it is monotonic decreasing (increasing); etc. The known behavior of the $X_{2,n}$ at infinity enables us to complete the theorem. It turns out (rather surprisingly) that the $X_{2,n}$ behave like $U = U\,(v; \alpha, n)$.

We begin, then, by putting $\frac{dZ}{dv}$ equal to zero in Eq. (6.7) and find that the explicit expressions for U and L are

$$U,L = \frac{1}{v}\left[-\frac{1}{v}(ns + 1) \pm \sqrt{1 + \frac{n^2+1}{v^2} + \frac{n}{s}\left[\frac{1}{\alpha^2} + \frac{2}{v^2}\right]}\;\right] \quad , \qquad (6.8)$$

where $s = \sqrt{1 + (\frac{v}{\alpha})^2}$; the plus sign goes with U, and the minus sign with L. It is clear from Eq. (6.8) that U and L are always real valued so that their graphs actually do divide the right half-plane into the three regions described above. (See Fig. 7)

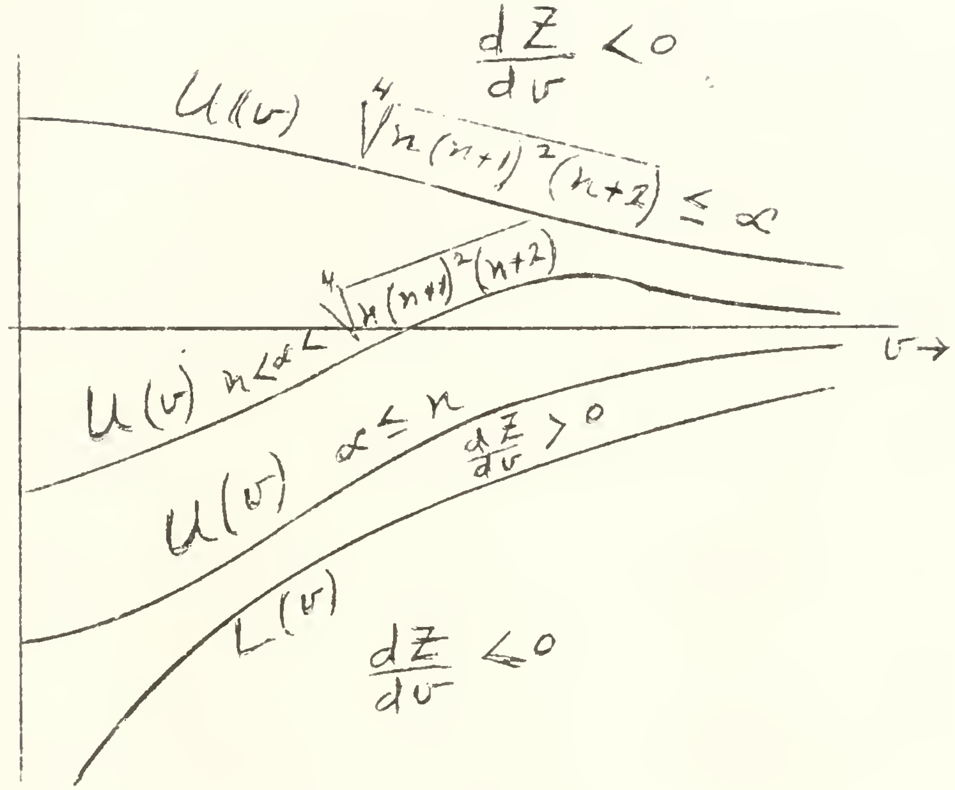


FIG 7

One sees from Eq. (6.8) that L(v) is the sum of negative monotonic increasing terms and hence is itself negative monotonic increasing. Furthermore, L(v) has a pole at the origin. It follows from Eq. (6.3) that the graph of $X_{2,n}$ starts out above that of L(v). Furthermore if it should ever touch the L(v)-curve it would cross (having a zero slope at this point) and enter a region of negative slope. It could never again intersect the L(v)-curve since L(v) is a monotonic increasing function. The value of $X_{2,n}$ at this point of intersection would thereafter be an upper bound for $X_{2,n}$. Since this value is necessarily less than zero, this would be contrary to the fact that $X_{2,n}$ approaches zero asymptotically as $v \rightarrow \infty$. Consequently the graph of $X_{2,n}$ lies above that of L(v) for all $v \geq 0$.

The quadratic equation in Z obtained from Eq. (6.7) by setting $\frac{dZ}{dv}$ equal to zero can be rewritten as a cubic function of s* . The result is

$$F(s) = \alpha^2 Z^2 s^3 + 2n Z x^2 - \left[\alpha^2 Z^2 + 1 - 2Z - \frac{n^2}{\alpha^2}\right] s + \frac{n}{\alpha^2} = 0 . \qquad (6.9)$$

To a given value of Z and a root $s \geq 1$, there is a $v \geq 0$ such that either $U(v) = Z$ or $L(v) = Z$. (This value of Z need not of course be assumed by the function $X_{2,n}$.) Since $F(-\infty) = -\infty$, $F(o) = \frac{n}{\alpha^2}$, and $F(\infty) = \infty$, it follows that F(s) has at most two roots for which $s \geq 1$. Since L(v) takes on all negative values, it follows that for $Z < 0$, one of these roots and only one necessarily corresponds to a point on L(v). Hence in the region that U(v) is negative it must be monotonic. On the other hand for positive Z, either zero, one, or two roots can lie on U(v). Hence in the region that U(v) is positive it is either monotonic or has a single maximum and minimum (since, as is readily shown, the graph of U(v) approaches the v-axis asymptotically as $v \to \infty$).

To conclude our description of U(v) we make use of the series and asymptotic expansions of U(v) which are readily obtainable from Eq. (6.8).

$$U(v) = \left\{\frac{1}{2(n+1)} - \frac{n}{2\alpha^2}\right\} - \left\{\frac{1}{8(n+1)^3} - \frac{n(n+2)}{8(n+1)\alpha^4}\right\} v^2$$

$$+ \left\{\frac{1}{16(n+1)^5} - \frac{n}{16(n+1)^3\alpha^4} - \frac{n(n+3)}{16(n+1)\alpha^6}\right\} v^4 + \ldots \qquad (6.10)$$

$$U(v) = \left(1 - \frac{n}{\alpha}\right)\frac{1}{v} - \left(1 - \frac{n}{2}\right)\frac{1}{v^2} + \ldots \qquad (6.11)$$

For $\alpha \leq n$, U(v) is negative for v near zero and also for sufficiently large v; thus U(v) is always negative and monotonic increasing. For $n < \alpha < \sqrt[4]{n(n+1)^2(n+2)}$, U(v) has a positive slope for small v and is positive as $v \to \infty$, which implies that it has a single maximum and no minima. Finally for $\alpha \geq \sqrt[4]{n(n+1)^2(n+2)}$, U(v) has a negative slope for small values of v and is positive as $v \to \infty$, and therefore is always positive and monotonic decreasing.

---

* The change of variable $s^2 = 1 + (\frac{v}{\alpha})^2$ establishes a 1 - 1 relationship between the two half-planes $s \geq 1$ and $v \geq 0$. The strip $0 \leq s < 1$ does not, of course, correspond to any part of the real v-plane. Curves in the s-half-plane go over into curves in the v-half-plane under a horizontal stretching process; the property of being monotonic and the property of having a horizontal tangent are preserved by the transformation. It is for this reason that we shall translate freely these properties proved in the s-variable into the v-variable without further mention.

We now return to the function $X_{2,n}$. A comparison of the series expansion for $X_{2,n}$ and U (Eqs. (6.5) and (6.10) shows that the graph of $X_{2,n}$ starts below that of U(v), for $\alpha < \sqrt[4]{n(n+1)^2(n+2)}$. Now the graph of $X_{2,n}$ cannot intersect that of U(v) from below at a point at which U(v) is increasing because at such a point the slope of $X_{2,n}$ would be zero (by definition of U). For $\alpha \leq n$, U is always monotonic increasing so that $X_{2,n}$ remains between L and U, that is in the positive slope region. Consequently in this case $X_{2,n}$ is negative and monotonic increasing.

For $n < \alpha < \sqrt[4]{n(n+1)^2(n+2)}$, $X_{2,n}$ can intersect U only at a point at which U has a non-positive slope. At such a point it must necessarily cross to the region above U since U is either decreasing or at its maximum whereas $X_{2,n}$ has a zero slope and can only decrease by crossing over to the region above U. Further, it is clear that $X_{2,n}$ must cross into this region since it approaches the v-axis from above and hence eventually is decreasing. Once in the region above U, $X_{2,n}$ must remain there for all larger v since U is thereafter a monotonic decreasing function and $X_{2,n}$ would necessarily have a zero slope at any point of intersection. Thus $X_{2,n}$ is increasing at the start, has its maximum at the point of intersection with U and is thereafter decreasing, approaching the v-axis asymptotically from above.

Finally, for $\alpha > \sqrt[4]{n(n+1)^2(n+2)}$, $X_{2,n}$ is greater than U at the start and since U is monotonic decreasing for $v > 0$, it follows, as above, that $X_{2,n}$ must remain above U for all $v > 0$. Hence in this case $X_{2,n}$ is positive and monotonic decreasing. This is likewise true when the inequality is replaced by an equality since the functions $X_{2,n}$ are continuous in . This concludes the proof of theorem 6.

Upper and lower bounds on the functions x_n and x_n' can now be deduced. If $\alpha^2 = n(n+1)$, the function $X_{2,n} = 0$ for v = 0 and is thereafter positive; for $\alpha = n$, $X_{2,n}$ is always negative. If $\alpha = \sqrt[4]{n(n+1)^2(n+2)}$, then $\frac{dX_{2,n}}{dv}$ is negative for $v > 0$; for $\alpha = n$, $\frac{dX_{2,n}}{dv}$ is positive for $v > 0$. These results are summarized in terms of $\varphi_n(v,\alpha)$, defined in section 4, $\varphi_n(v,\alpha) = \frac{n}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}}$, (4.10)

by means of the following corollary:

Corollary $\quad \varphi_n\left(v, \sqrt{n(n+1)}\right) < x_n < \varphi_n(v,n)$,

$$\varphi_n'(v,n) < x_n' < \varphi_n'\left(v, \sqrt[4]{n(n+1)^2(n+2)}\right),$$

for all $v > 0$ and $n \geq 1$.

7. The Functions $Y_{2,n}$.

In this section we shall study the functions $Y_{2,n}(v)$ where

$$Y_{2,n}(v) = Y_{2,n} = -\frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} + y_n \ . \tag{7.1}$$

We shall prove theorems similar to theorems 5 and 6 of section 6. The proof of theorem 8, which characterizes the functions $Y_{2,n}(v)$, for all values of the parameters n and α , is the most difficult proof in the present paper.

$$\text{Let} \quad \bar{Y}_n = y_n - \frac{n}{v^2} \tag{7.2}$$

Equations (5.2), (5.4) and (5.6) yield the following equations for $\bar{Y}_n$:

$$\frac{d\bar{Y}_n}{dv} = \bar{Y}_n^2 v + \frac{2}{v}(n-1)\bar{Y} - \frac{1}{v} \ , \tag{7.3}$$

$$\left.\begin{aligned} \bar{Y}_n &= \frac{1}{2(n-1)} - \frac{v^2}{8(n-1)^2(n-2)} + \ldots., \quad n > 1 , \\ \bar{Y}_1 &= -\log\frac{\gamma v}{2} + \ldots. \\ & \qquad\qquad\qquad\qquad v \text{ small} \\ \bar{Y}_0 &= y_0 = -\frac{1}{v^2 \log\frac{\gamma v}{2}} + \ldots. \end{aligned}\right\} \tag{7.4}$$

$$\bar{Y}_n = \frac{1}{v} - \frac{1}{2}\,\frac{2n-1}{v^2} + \ldots. \ , \quad n \geq 0,\ v \text{ large} \tag{7.5}$$

As in section 6, the ordering theorem is again a result of the two following lemmas:

Lemma 7. $\bar{Y}_n$ <u>is a positive monotonic decreasing function of</u> v <u>for all</u> $v > 0$.

Lemma 8. $\bar{Y}_n > \bar{Y}_{n+1}$ for all $n \geq 0$, $v \geq 0$.

The proof of lemma 7 is similar to that of lemma 3. We shall give only an outline of the proof. Equqtion (7.4) shows that the graph of $\bar{Y}_n$ starts out above the v-axis. Further, the supposition that it crosses the v-axis leads to

$\frac{d\bar{Y}_n}{dv} = -\frac{1}{v} < 0$ at the point of crossing. This is inconsistent with the fact that $\bar{Y}_n$ approaches zero from above as v tends to ∞. Hence $\bar{Y}_n$ is positive for all $v > 0$.

Again, $\frac{d\overline{Y}_n}{dv}$ is negative for v near zero and also for large v. If it ever became zero, say at $v = v_1$, then it would follow that $\frac{d^2\overline{Y}_n}{dv^2} = 2\overline{Y}_n^2 > 0$ at $v = v_1$, which implies that $\frac{d\overline{Y}_n}{dv}$ would remain positive thereafter. This is inconsistent with the behavior of $\frac{d\overline{Y}_n}{dv}$ for large v. The lemma is therefore proved.

If we put $\Delta\overline{Y} = \overline{Y}_{n+1} - \overline{Y}_n$, it follows from Eq. (7.5) that $\overline{Y}$ is negative for large v. The differential equation for $\overline{Y}$ is

$$\frac{d(\Delta\overline{Y})}{dv} = v(\overline{Y}_n + \overline{Y}_{n+1})\Delta\overline{Y} + \frac{2n}{v}\Delta\overline{Y} + \frac{2}{v}\overline{Y}_n$$

If $\overline{Y}$ were ever zero, let $v = v_1$ be the largest v for which this is true. Then $\frac{d(\Delta\overline{Y})}{dv} = \frac{2}{v}\overline{Y}_n > 0$, which implies that $\Delta\overline{Y}$ would be positive thereafter. This contradicts the known behavior of $\Delta\overline{Y}$ for large v, and hence lemma 8 is proved.

Let $\Delta Y = Y_{2,n+1} - Y_{2,n}$.

Then from Eq. (7.1) and the definition of $\Delta\overline{Y}$, we can write ΔY as

$$\Delta Y = \frac{1}{v^2}\left\{1 - \left[1 + \frac{v^2}{\alpha^2}\right]^{-1/2}\right\} + \Delta\overline{Y} \quad .$$

Hence ΔY is the sum of two negative functions. This establishes

<u>Theorem 7</u>. $Y_{2,n} > Y_{2,n+1}$ <u>for all</u> $n \geq 0$ <u>and</u> $v \geq 0$.

We now proceed to

<u>Theorem 8</u>. <u>If</u> $\alpha \leq \sqrt[4]{n(n-1)^2(n-2)}$, <u>then</u> $Y_{2,n}$ <u>is negative and monotonic increasing, and its graph approaches the v-axis from below; if</u> $\sqrt[4]{n(n-1)^2(n-2)} < \alpha < n$, <u>then</u> $Y_{2,n}$ <u>has a single minimum and no maxima and its graph thereafter approaches the v-axis from below; if</u> $n \geq \alpha$, <u>then</u> $Y_{2,n}$ <u>is positive and monotonic decreasing, and its graph approaches the v-axis from above</u>. This theorem is valid for all $n \geq 0$ and $v > 0$.

The theorem is illustrated by the graph in Fig. 8.

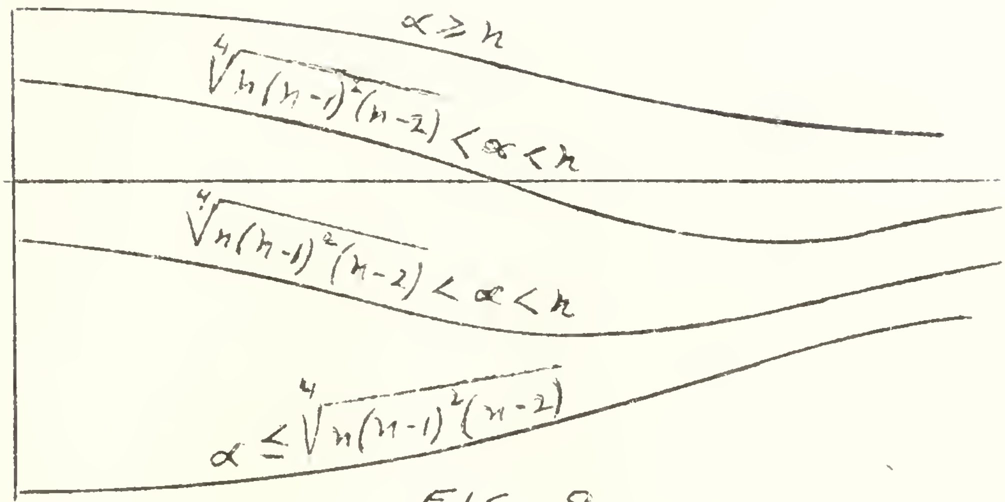


FIG 8

The case n = 0 is included in lemma 3, and will be omitted in the following.

The present theorem is more difficult to prove than theorem 6; the general idea of the proof, however, is the same in both cases.

The series and asymptotic expansions of $Y_{2,n}$ are:

$$\begin{cases} Y_{2,1} = -\log \dfrac{v}{2} - \dfrac{1}{2\alpha^2} + \dots \\ Y_{2,n} = \dfrac{1}{2(n-1)} - \dfrac{n}{2\alpha^2} - \dfrac{1}{8}\left\{ \dfrac{1}{(n-1)^2\,(n-2)} - \dfrac{n}{\alpha^4} \right\} v^2 + \dots \quad (n>0) \end{cases}$$

for small v (7.6)

and

$$Y_{2,n} = \left(1 - \frac{n}{\alpha}\right)\frac{1}{v} + \frac{1}{2v^2} + \dots \quad (n \geq 0)$$

for large v. (7.7)

In addition the function $Y_{2,n}$ satisfies the following differential equation:

$$\frac{dZ}{dv} = vZ^2 + \frac{2}{v}\left\{ n\left[1 + \frac{v^2}{\alpha^2}\right]^{1/2} - 1 \right\} Z - \frac{1}{v} + \frac{1}{v}\,\frac{n^2}{\alpha^2} - \frac{n}{v}\,\frac{1}{\alpha^2}\,\frac{1}{\left[1 + \frac{v^2}{\alpha^2}\right]^{1/2}} \quad (7.8)$$

As in theorem 6, we are interested in the locus of points where $\frac{dZ}{dv} = 0$. This locus is obtained as the solution of a quadratic in Z and defines the two functions U(v) and L(v) which are respectively the greater and the smaller of the roots of this quadratic. Putting $\frac{dZ}{dv} = 0$ in Eq. (7.8) we find explicit expressions

for U and L. They are

$$U, L = \frac{1}{v^2}(1 - ns) \pm \frac{1}{v}\sqrt{1 + \frac{1+n^2}{v^2} - \frac{n}{s}\left(\frac{1}{\alpha^2} + \frac{2}{v^2}\right)}\,, \qquad (7.9)$$

where $s = \sqrt{1 + \frac{v^2}{\alpha^2}}$; the plus sign goes with U and the minus sign with L.

It is no longer true that U and L are defined for all v, and n. One can therefore not expect U and L to have the simple properties that they had in theorem 6; it is precisely this fact that complicates the proof of the present theorem. If U and L are everywhere defined, the half-plane $v > 0$ breaks up into three regions in each of which the sign of $\frac{dZ}{dv}$ is everywhere the same: the region above the U-curve in which $\frac{dZ}{dv}$ is positive, the region between the U- and L-curves in which $\frac{dZ}{dv}$ is negative, and the region below the L-curve in which $\frac{dZ}{dv}$ is positive. When U and L are not everywhere defined, the U- and L-curves have two branches as indicated in Fig. 9; in the regions interior to the U,L-curves $\frac{dZ}{dv}$ is negative, in the region exterior to the U,L-curve, $\frac{dZ}{dv}$ is positive.

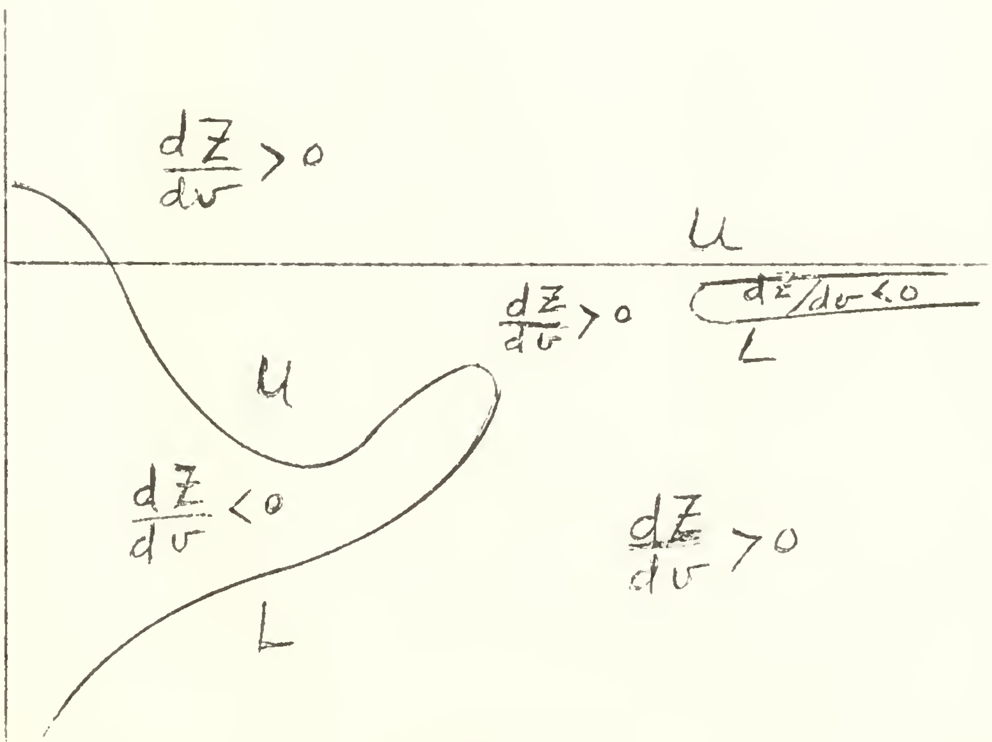


FIG. 9

One can readily obtain the series and asymptotic expansions of U from Eq. (7.9). They are

$$\begin{cases} U_1 = \frac{1}{v} - \frac{1}{2\alpha^2} + \dots \\ U_n = \frac{1}{2(n-1)} - \frac{n}{2\alpha^2} - \frac{v^2}{8(n-1)}\left[\frac{1}{(n-1)^2} - \frac{n(n-2)}{\alpha^4}\right] + \dots (n>1) \end{cases}$$

for small v , (7.10)

and

$$U_n = (1 - \frac{n}{\alpha})\frac{1}{v} + (1 - \frac{n}{2\alpha})\frac{1}{v^2} + \dots \quad (n \geq 1) \quad . \tag{7.11}$$

for large v.

We see from Eq. (7.9) that for those values of v for which L exists, L is the sum of two negative terms and is therefore itself negative. Furthermore, L has a pole at the origin. It follows that any curve which is bounded from below at the origin (hence $Y_{2,n}$ in particular) starts out above the L-curve.

As in theorem 6, we shall again study the U- and L-curves by means of the cubic function of s obtained from Eq. (7.8) by setting $\frac{dZ}{dv} = 0$, namely

$$F(s) = \alpha^2 Z^2 s^3 + 2nZs^2 - \left\{\alpha^2 Z^2 + 2Z + 1 - \frac{n^2}{\alpha^2}\right\} s - \frac{n}{\alpha^2} \tag{7.12}$$

To each value of Z and a root $s \geq 1$ of $F(s) = 0$, there is a v such that either $U(v) = Z$ or $L(v) = Z$.

We shall first consider the case $\alpha \geq n$. It is easy to see that the radicand in Eq. (7.9) is positive since

$$1 > \frac{1}{sn} \geq \frac{n}{s\alpha^2} \text{ and } 1 + n^2 \geq 2n > \frac{2n}{s} \text{ for } s > 1.$$

It follows that U and L are defined (and single valued) for all $s \geq 1$. Furthermore, for $Z > 0$ and $\alpha \geq n$ we see that the coefficients of F(s) have the signs (+ + - -) in the order of decreasing powers of s. Thus there is just one variation in sign, and by Descartes' rule of signs there is precisely one root for s > 0. Now the U-curve has a positive ordinate and negative slope at s = 1 and approaches the v-axis from above as $v \to \infty$. If U were to have a minimum there would be values of Z such that U(v) = Z for at least two different values of v, and hence two different positive values of s. Thus, F(s) = 0 for more than one positive value of s. Since this is impossible, U can have no minimum and we conclude that U is positive and monotonic decreasing for $\alpha > n$.

From the expansion of $Y_{2,n}$ (Eq. (7.6)) we see that the graph of $Y_{2,n}$ starts off with a negative slope and hence lies between the U-curve and the L-curve for small $v > 0$. Since U is monotonic decreasing, the graph of $Y_{2,n}$ could cross that of U with zero slope. If it did so, however, it would thereafter be in the region of positive slope above the graph of $U(> 0)$, and hence would be bounded away from zero. This is contrary to the fact that $Y_{2,n}$ approaches zero from above as v approaches infinity (see Eq. (7.7)). Finally if $Y_{2,n}$ were ever zero, the asymptotic behavior of $Y_{2,n}$ indicates that it would be zero an even number of times. This means that, at the points of crossing, its slope would alternate in sign. This is impossible since for $Y_{2,n} = Z = 0$, Eq. (7.8) reduces to

$$\frac{dZ}{dv} = -\frac{1}{v}\left[1 - \frac{n^2}{\alpha^2}\right] - \frac{n}{v}\frac{1}{\alpha^2}\frac{1}{\sqrt{1 + \frac{v^2}{\alpha^2}}} < 0 \quad \text{for all } v > 0.$$

It follows that $Y_{2,n}$ is positive and monotonic decreasing for $\alpha \geq n$.

We have as a corollary to the first part of the theorem that

$$y_n > \frac{n}{v^2}\sqrt{1 + \frac{v^2}{n^2}} \quad . \tag{7.13}$$

for all $v \geq 0$. Hence if we define the auxiliary function

$$\psi(v) = \frac{n}{v^2}\left[\sqrt{1 + \frac{v^2}{n^2}} - \sqrt{1 + \frac{v^2}{\alpha^2}}\right], \tag{7.14}$$

it follows from (7.13) that

$$Y_{2,n} > \psi \qquad \text{for all } v \geq 0. \tag{7.15}$$

We now consider the case $\alpha < n$. In this case U and L need not be defined for all $v \geq 0$. It will be convenient to consider U and L* as functions of s; that is, as roots of $F(s) = 0$.

$$U,L = \frac{1 - ns}{\alpha^2(s^2 - 1)} \pm \frac{1}{\alpha^2(s^2-1)}\sqrt{1 + n^2 + \alpha^2(s^2-1) - \frac{n}{s}(1 + s^2)} \quad .$$

---

* It should be noted that for $0 < s < 1$, L can be positive; whereas for $1 \geq s$ (i.e. $v > 0$) L must be negative where it exists.

For s small and positive, the radicand is negative so that the curves do not exist in this region. Each curve has its only positive singularity at $s = 1$. This can be seen from the expansion of U about $s = 1$

$$U = \frac{1-n}{\alpha^2(s-1)} + \dots \qquad s < 1,\ n > 1$$

$$U = \frac{1}{\alpha\sqrt{2(s-1)}} + \dots \qquad s > 1,\ n = 1$$

Now for $Z > 0$ the coefficients of $F(s)$ have the signs $(+\ +\ \pm\ -)$ and hence $F(s)$ has one and only one positive real root. If $Z = 0$, $F(s) = 0$ for $s = \dfrac{1}{n(1-\sigma)}$, where we have set $\sigma = \dfrac{\alpha^2}{n^2}$. For any $Z \geq 0$ the value of s satisfying $F(s) = 0$ makes $U(s) = Z$. Hence, the graph of $Z = U(s)$ crosses the s-axis at $P_1 : \left(\dfrac{1}{n(1-\sigma)}, 0\right)$ and crosses every line $Z = \text{const.} > 0$ just once. If we try to trace the curve $Z = U(s)$ we find first that for $n > 1$ it approaches the line $s = 1$ asymptotically from the left; it cannot cross the Z-axis, so that it must double back on itself; since it crosses each horizontal line $(Z > 0)$ just once it can have no minimum; $F(s)$ is quadratic in Z and hence the curve cannot double back on itself more than once for $s < 1$; for $s \geq 1$ it cannot double back on itself above the s-axis since this would imply a root $L > 0$ which is impossible for $s \geq 1$. It follows that for $Z > 0$ the curve must look like the sketches in Fig.10a and 10b.

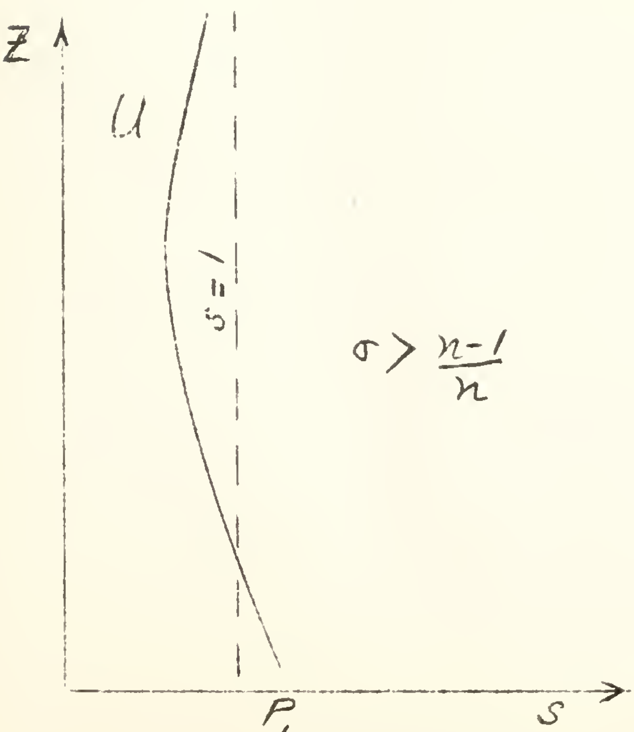


Fig. 10a

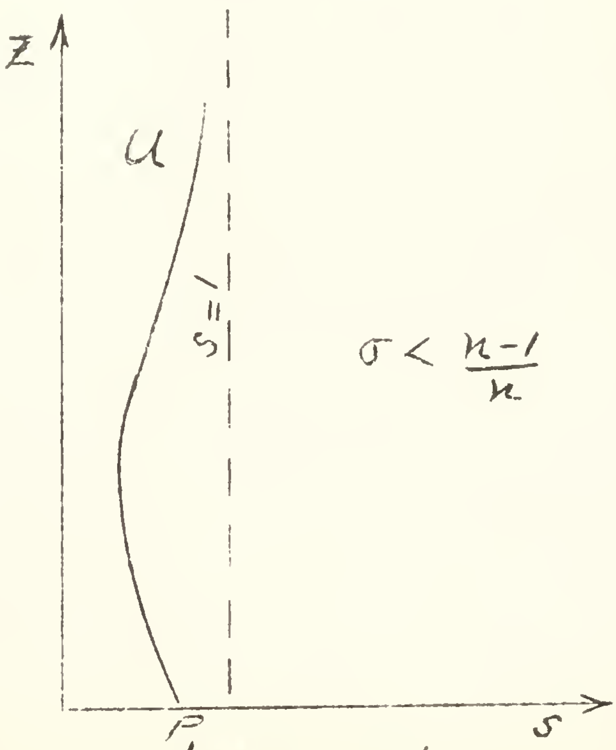


Fig. 10b

If $n = 1$, the graph of $Z = U$ approaches the line $s = 1$ asymptotically from the right, and since it crosses the s-axis at $\frac{1}{1-\sigma} > 1$, it must decrease monotonically until it reaches this point. (See Fig. 10c)

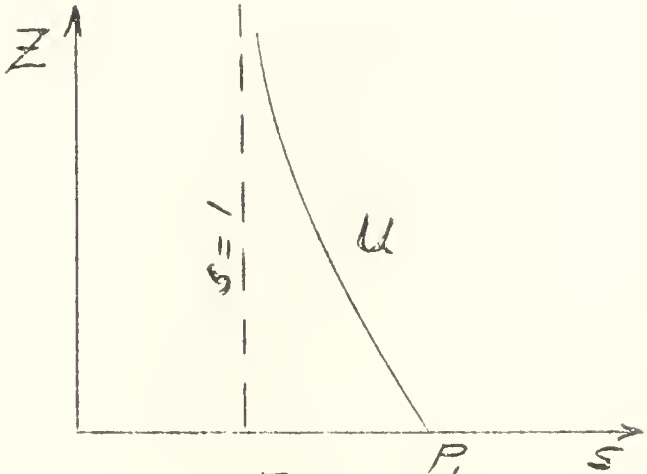


FIG. 10c

We shall study U for $Z < 0$ by means of the auxiliary function defined by Eq. (7.14). At $v = 0$, $U = \frac{1}{2(n-1)} - \frac{n}{2\alpha^2}$, $\psi = \frac{1}{2n} - \frac{n}{2\alpha^2}$, and $L = -\frac{1}{2(n-1)} - \frac{n}{2\alpha^2}$; therefore lies between U and L at $v = 0$. In order to determine the points at which the graph of intersects that of U or L we go back to the equation defining U and L; that is we insert $Z = \psi(v)$ in the equation $\frac{dZ}{dv} = 0$. Setting $\frac{dZ}{dv} = 0$, one obtains by direct substitution of (7.14) into (7.8)

$$\frac{2n}{v^3}\left\{\sqrt{1+\frac{v^2}{\alpha^2}} - \sqrt{1+\frac{v^2}{n^2}}\right\} - \frac{n}{v\alpha^2}\,\frac{1}{\sqrt{1+\frac{v^2}{\alpha^2}}} = 0 .$$

Solving for the real roots in v (that is $s \geq 1$), we get

$$\frac{v^2}{\alpha^2} = \frac{4\sigma}{1-4\sigma} \qquad \left(\sigma = \frac{\alpha^2}{n^2} > 0\right)$$

if and only if $\sigma < \frac{1}{4}$. For this value of v,

$$\psi = -\frac{n}{2\alpha^2}\sqrt{1-4\sigma} \quad \text{and} \quad s = \frac{1}{\sqrt{1-4\sigma}} .$$

It follows that for $\frac{n}{2} \leq \alpha < n$, the graph of Ψ intersects neither the U-nor the L- curves for $v \geq 0$, since in this case $\sigma \geq \frac{1}{4}$. This implies that U and L are not connected and hence must exist and be single-valued for all $v > 0$. The graph of L starts at minus infinity and approaches the v-axis asymptotically from below. Hence L must assume all negative values of Z at least once. For a given value of $Z < 0$, U can therefore correspond to at most two roots of the cubic $F(s) = 0$. Since the U-curve crosses into the region $Z < 0$ from above and approaches the s-axis asymptotically from below, U can have but a single minimum for α in the range $n/2 \geq \alpha < n$. It is clear that $U > \Psi > L$. The situation is therefore represented by the sketches in Fig. 11.*

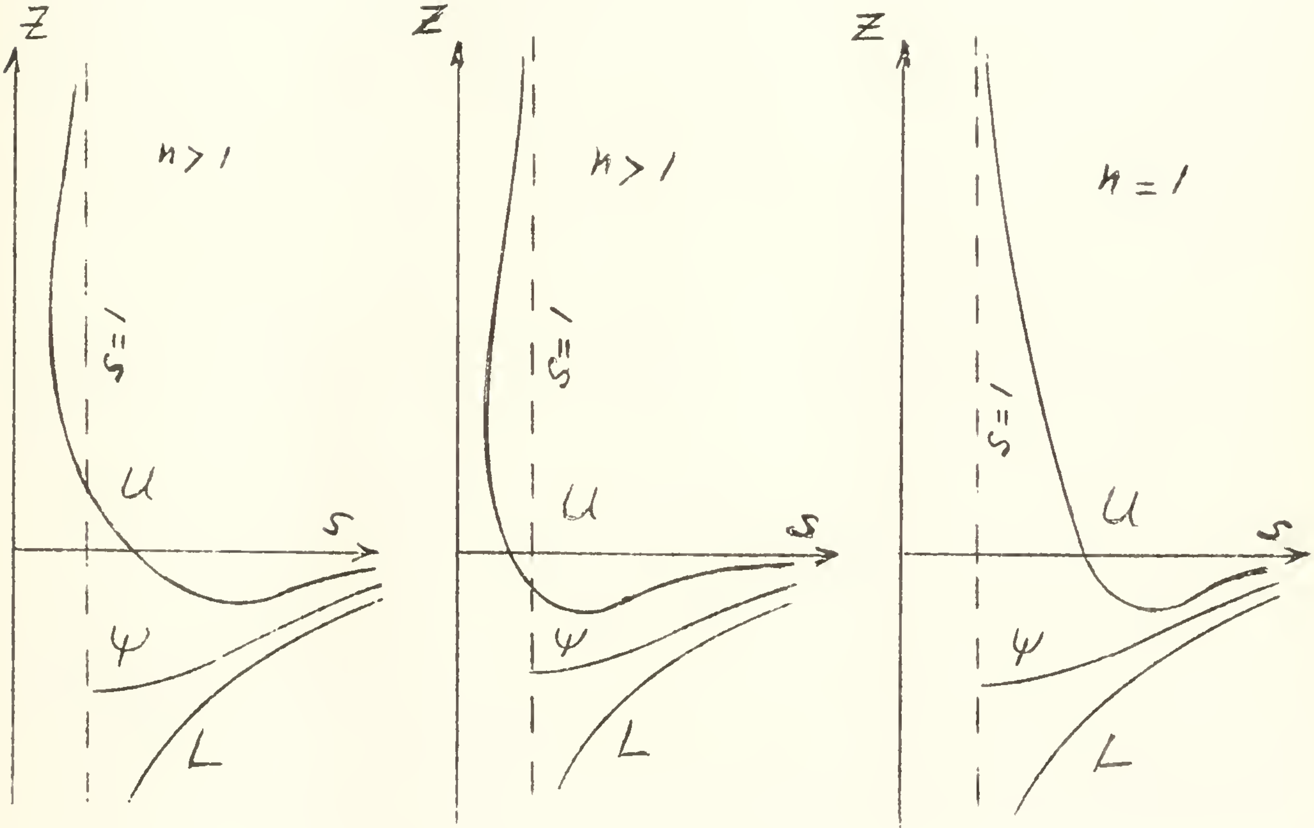


FIG. 11

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* It should be noted that for $s \geq 1$, U is single-valued because of the existence of L below it. For $s < 1$, however, the extension of v to the left of $s = 1$ (the lower part of which should properly be called L) is single-valued because of the existence of the upper branch of U which approaches $s = 1$ asymptotically from the left.

For $\alpha < \frac{n}{2}$, the function ψ has just one point in common with U or L $(v \geq 0)$, at $P_2 : \left(\frac{1}{\sqrt{1-4\sigma}}, -\frac{1}{2\sigma n}\sqrt{1-4\sigma}\right)$, in the (sZ)-plane. Now

$$\psi - \frac{U+L}{2} = \frac{n\sqrt{1+\frac{v^2}{n^2}} - 1}{v^2} > 0 ;$$

thus ψ is always greater than the average of U and L $(v \geq 0)$, and hence the graph of ψ certainly lies above that of L. Therefore it intersects the graph of U once $(v \geq 0)$ and thereafter lies above both U- and L- curves.

We wish to show that to the left of P_2, that is, for $0 < s \leq \frac{1}{\sqrt{1-4\sigma}}$, U has at most one minimum and no maximum. We show first that in this range F(s) has precisely one root for $-\frac{1}{2n\sigma}\sqrt{1-4\sigma} < Z < 0$. It follows that there is one value of s for each Z in this range for which $U(s) = Z$. We define G(Z) as follows:

$$F\left(\frac{1}{\sqrt{1-4\sigma}}\right) = \left(\frac{4\sigma^2 n^2}{(1-4\sigma)^{3/2}}\right) Z^2 + 2\,\frac{n-\sqrt{1-4\sigma}}{1-4\sigma}\,Z - \frac{1-\sigma-\frac{\sqrt{1-4\sigma}}{n}}{\sigma\sqrt{1-4\sigma}} \equiv G(Z) .$$

G(Z) is thus a parabolic function of Z, and

$$G(o) = \frac{1-\sigma-\frac{\sqrt{1-4\sigma}}{n}}{\sigma\ (1-\sigma)} > 0$$

since $\frac{\sqrt{1-4\sigma}}{n} \leq \sqrt{1-4\sigma} < 1-2\sigma < 1-\sigma$.

Since P_2 lies on U, $G\left(-\frac{\sqrt{1-4\sigma}}{2n\sigma}\right) = 0$. The minimum of the parabolic function, G(Z), is

$$Z_m = -\frac{\sqrt{1-4\sigma}}{2n\sigma} \cdot \frac{n-\sqrt{1-4\sigma}}{2n\sigma} < -\frac{\sqrt{1-4\sigma}}{2n\sigma} ,$$

since $n-\sqrt{1-4\sigma} \geq n(1-\sqrt{1-4\sigma}) > 2n\sigma$. Hence Z_m lies below P_2. Now $G(Z)=0$ for $Z = -\frac{\sqrt{1-4\sigma}}{2n}$ and, since the minimum of G(Z) occurs for $Z < -\frac{\sqrt{1-4\sigma}}{2n\sigma}$ and, $G(0) > 0$, it follows that G(Z) is greater than 0 for all Z in the range $\frac{-\sqrt{1-4\sigma}}{2n\sigma} < Z < 0$.

Returning to the function F(s), we have

$$F(0) = -\frac{n^2}{\alpha^2} < 0, \text{ Z arbitrary;}$$

$$F\left(\frac{1}{\sqrt{1-4\sigma}}\right) \equiv G(Z) > 0 \text{ for } -\frac{\sqrt{1-4\sigma}}{2n\sigma} < Z < 0.$$

Hence the equation $F(s) = 0$ has an odd number of roots for $0 \leq s \leq \frac{1}{\sqrt{1-4\sigma}}$ and $-\frac{\sqrt{1-4\sigma}}{2n\sigma} < Z < 0$. If we can show that the graph of F(s) has no point of inflection in this region it will follow that there is exactly one root. Now $F''(s)=0$ for

$$s = -\frac{2}{3n\sigma Z}.$$

For $Z > \frac{\sqrt{1-4\sigma}}{-2n\sigma}$, i.e. $-Z < \frac{\sqrt{1-4\sigma}}{2n\sigma}$,
this value of $s > \frac{4}{3\sqrt{1-4\sigma}} > \frac{1}{\sqrt{1-4\sigma}}$; thus the point of inflection is to the right of P_2.

As we have seen the U-curve enters the lower half of the (s,Z)-plane by crossing the s-axis at the point P_1. Since F(s) has precisely one root to the left of P_2 for each negative Z above P_2, the U-curve must proceed downward without a maximum or a minimum in the region R above and to the left of P_2. It must pass out of R at P_2 or to the left of P_2, since P_2 would otherwise lie on L. If it passes out of R to the left of P_2 it can not again enter R because if it were to do so F(s) would have more than one root for some value of Z in R. On the other hand, the curve eventually passes through P_2 so that it must have a minimum below P_2. Furthermore it can have no other extremum below P_2 since F(s) can have at most three roots for any value of Z. Finally for v in the range $0 \leq v \leq 2\alpha\sqrt{\frac{\sigma}{1-4\sigma}}$ (that is $1 \leq s \lesssim \frac{1}{\sqrt{1-4\sigma}}$), the curve L exists below ψ ; it follows that U(b) is single-valued in this range. Thus U(v) is single-valued and has at most one minimum but no maxima to the left of P_2. There are thus only three possible

descriptions of $U(v)$ to the left of P_2. $U(v)$ is monotonic decreasing; $U(v)$ has a single minimum and no maximum; $U(v)$ is monotonic increasing. The latter case occurs only if $U(v)$ is initially increasing, i.e. if $\alpha \leq \sqrt[4]{n(n-1)^2(n-2)}$. The possible kinds of U-curves are sketched in Fig. 12.

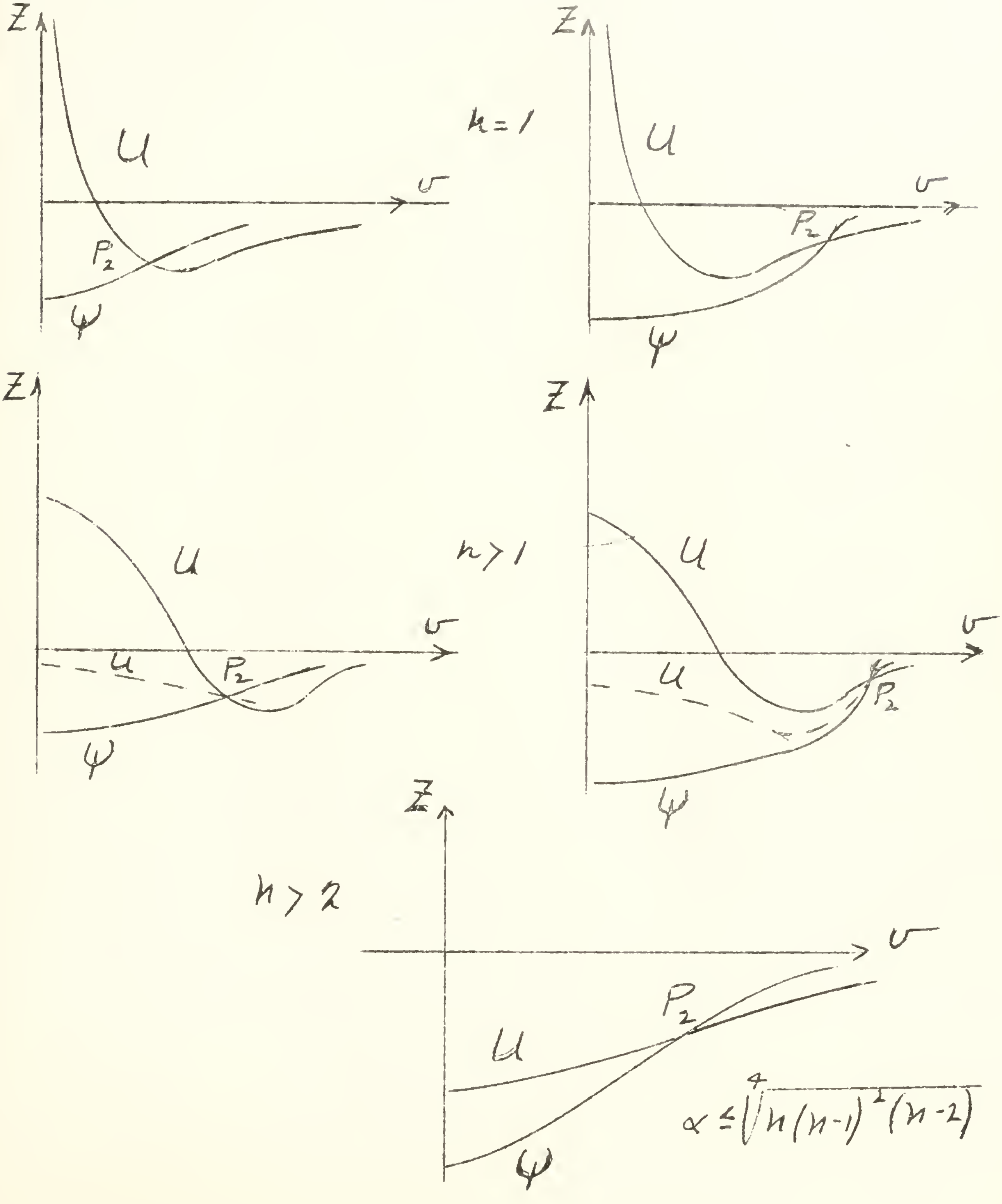


FIG. 12

The proof of the theorem for $\alpha < n$ is now straightforward. When $\sqrt[4]{n(n-1)^2(n-2)} < \alpha < n$, the graph of $Y_{2,n}$ starts off below the U-curve with a negative slope. In the upper half-plane U decreases monotonically until its graph intersects the v-axis. The graph of $Y_{2,n}$ cannot intersect that of U above the v-axis, for if it did so it would enter a region of positive slope and remain bounded away from the v-axis, this is contrary to the behavior of $Y_{2,n}$ at infinity. Furthermore, the graph of $Y_{2,n}$ cannot intersect that of U (with zero zlope) where U is increasing. On the other hand, since $Y_{2,n}$ is eventually increasing, it must enter a region of positive slope at some time. Now $Y_{2,n} > \psi > L$. Hence the graph of $Y_{2,n}$ intersects the U-curve below the v-axis at a point at which U is decreasing. Thereafter $Y_{2,n}$ cannot enter a region of negative slope. For if it intersected the graph of U again, the intersection would have to occur at a point of increasing slope for U to the left of P_2. In this case $Y_{2,n}$ would become trapped between ψ and a monotonically increasing portion of U; it could never again enter a region of increasing slope.

For $\alpha \leq \sqrt[4]{n(n-1)^2(n-2)}$, the graphs of both $Y_{2,n}$ and U start out with positive slopes and with $Y_{2,n}$ above U. In this case U is monotonic increasing at least up to the point P_2. Hence if $Y_{2,n}$ ever entered a region of decreasing slope, it would have to intersect the U-curve to the left of P_2. As before $Y_{2,n}$ would thereafter be trapped between ψ and a monotonically increasing portion of U. In this case $Y_{2,n}$ would remain in a region of decreasing slope and therefore be bounded away from the v-axis; this is contrary to the behavior of $Y_{2,n}$ at infinity. It follows that $Y_{2,n}$ is monotonic increasing for all $v > 0$. This concludes the proof of theorem 8.

If we make use of the function $\varphi_n(v,\alpha)$ defined by Eq. (4.10), that is

$$\varphi_n(v,\alpha) = \frac{n}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}}\ , \tag{4.10}$$

theorem 8 furnishes us with some useful bounds on the function, y_n and its derivatives. For $\alpha^2 = n(n-1)$, $Y_{2,n}$ is initially zero and is negative thereafter; for $\alpha = n$, $Y_{2,n}$ is always positive. For $\alpha = \sqrt[4]{n(n-1)^2(n-2)}$, $\frac{dY_{2,n}}{dv}$ is positive for

$v > 0$; for $\alpha = n$, $\dfrac{dY_{2,n}}{dv}$ is negative for $v > 0$. We have thus proved

Corollary 1. $\varphi_n(v,n) < y_n < \varphi_n'(v, \sqrt{n(n-1)})$

$$\varphi_n'\left(v, \sqrt[4]{n(n-1)^2(n-2)}\right) < y_n' < \varphi_n'(v,n)$$

for all $n \geq 1$ and all $v > 0$.

Since $y_n - \dfrac{n}{v^2}\sqrt{1 + \dfrac{v^2}{n^2}} > 0$, $x_n - \dfrac{n}{v^2}\sqrt{1 + \dfrac{v^2}{n^2}} < 0$,

we deduce

Corollary 2 $\quad y_n - x_n > 0$ for all $v \geq 0$ and $n \geq 1$.

Corollary 2 is likewise true for $n = 0$ (see the corollary to lemma 14 in section 8). Recalling the definitions of x_n and y_n, we see that

$$y_n - x_n = -\frac{1}{v}\left[\frac{K_n'(v)}{K_n(v)} + \frac{I_n'(v)}{I_n(v)}\right] = -\frac{1}{v}\,\frac{\left(K_n(v)\,I_n(v)\right)'}{K_n(v)\,I_n(v)} \quad .$$

Since $K_n(v)$ and $I_n(v)$ are positive for all $v > 0$, corollary 2 is equivalent to

$$(K_n\,I_n)' < 0,$$

from which we derive the interesting conclusion that the function $K_n\,I_n$ is a positive monotonic decreasing function.

8. Further Approximations.

The purpose of the present section is to find upper bounds for the functions $x_{n+1} - x_n$, $y_{n+1} - y_n$. These will be useful in proving that the functions $W_{2,n}$ are ordered with respect to n. The section ends with some inequalities on the particular functions x_0 y_0 and y_1.

Let

$$\widetilde{X}_n = x_n - \frac{n}{v^2} \cdot \frac{1}{\left(1 + \frac{v^2}{\alpha^2}\right)^{1/2}} \tag{8.1}$$

$$\Delta\widetilde{X} = \widetilde{X}_{n+1} - \widetilde{X}_n$$

It follows from lemma 5, section 6 that

$$\tilde{X}_n \geq \bar{X}_n > 0.$$

The differential equations and series expansions for $\tilde{X}_n$ and $\Delta\tilde{X}$ are easily obtainable from Eqs.(5.1) and (5.3). They are

$$\frac{d\tilde{X}_n}{dv} = -\tilde{X}_n^2 v - \frac{2n}{v}\frac{\tilde{X}_n}{\sqrt{1+\frac{v^2}{\alpha^2}}} - \frac{2}{v}\tilde{X}_n + \frac{1}{v} + \frac{n^2}{v^3}$$

$$+ \frac{n}{v}\frac{1}{\alpha^2}\frac{1}{(1+\frac{v^2}{\alpha^2})^{3/2}} - \frac{n^2}{v^3}\frac{1}{(1+\frac{v^2}{\alpha^2})}, \qquad (8.2)$$

$$\frac{d(\Delta\tilde{X})}{dv} = -v(\tilde{X}_{n+1}+\tilde{X}_n)\Delta\tilde{X} - \frac{2n}{v}\frac{\Delta\tilde{X}}{(1+\frac{v^2}{\alpha^2})^{1/2}} - \frac{2\tilde{X}_{n+1}}{v}\frac{1}{(1+\frac{v^2}{\alpha^2})^{1/2}}$$

$$-\frac{2}{v}\Delta\tilde{X} + \frac{2n+1}{v^3} + \frac{1}{v}\frac{1}{\alpha^2}\frac{1}{(1+\frac{v^2}{\alpha^2})^{3/2}} - \frac{2n+1}{v^3}\frac{1}{(1+\frac{v^2}{\alpha^2})},$$

and

$$\tilde{X}_n = \frac{1}{2}\left\{\frac{1}{n+1} + \frac{n}{\alpha^2}\right\} - \frac{v^2}{8}\left\{\frac{1}{(n+1)^2(n+2)} + \frac{3n}{\alpha^4}\right\} + \cdots,$$

$$\Delta\tilde{X}_n = \frac{1}{2}\left\{\frac{1}{\alpha^2} - \frac{1}{(n+1)(n+2)}\right\} + \frac{v^2}{8}\left[\frac{3n+5}{(n+1)^2(n+2)^2(n+3)} - \frac{3}{\alpha^4}\right] \qquad (8.3)$$

We now state the important lemma:

<u>Lemma 9</u>. $\Delta\tilde{X} < 0$ <u>if</u> $\alpha^2 \geq (n+1)(n+2)$ <u>for</u> $n \geq 0$, <u>and</u> $v > 0$.

Equation (8.3) shows that if $\alpha^2 \geq (n+1)(n+2)$, $\Delta\widetilde{X}$ is negative for small v. This implies that if $\Delta\widetilde{X}$ were ever zero for $v > 0$, its graph would cross the v-axis with non-negative slope. We shall therefore prove the lemma by showing that $\frac{d(\Delta\widetilde{X})}{dv}$ is negative whenever $\Delta\widetilde{X}$ is zero.

In Eq. (8.2) put $\Delta\widetilde{X}$ equal to zero, there results,

$$\frac{d(\Delta\widetilde{X})}{dv} = -2\,\frac{\widetilde{X}_{n+1}}{v}\cdot\frac{1}{s} + \frac{2n+1}{v^3}\;\frac{1}{v}\cdot\frac{1}{\alpha^2}\cdot\frac{1}{s^3} - \frac{2n+1}{v^3}\;\frac{1}{s^2}$$

where $s^2 = (1 + \frac{v^2}{\alpha^2})$. We wish to show that the right hand side of this equation is negative. That is to say

$$\widetilde{X}_{n+1} = X_{n+1} - \frac{n+1}{v^2}\cdot\frac{1}{s} > \frac{(2n+1)s}{2v^2} + \frac{1}{2\alpha^2 s^2} - \frac{2n+1}{2v^2 s} \quad . \tag{8.4}$$

But $x_{n+1} > \frac{n+1}{v^2}\sqrt{1 + \frac{v^2}{(n+1)(n+2)}}$ for $v > 0$ by the corollary to theorem 6. The inequality (8.4) is therefore valid if

$$\frac{(n+1)}{v^2}\left(1 + \frac{v^2}{(n+1)(n+2)}\right)^{1/2} \geq \frac{(2n+1)\;s}{2v^2} + \frac{1}{2\alpha^2 s^2} + \frac{1}{2v^2 s} \;,$$

$$\frac{(n+1)}{v^2}\left(1 + \frac{v^2}{(n+1)(n+2)}\right)^{1/2} \geq \frac{n+1}{v^2 s} + \frac{2n+1}{2\alpha^2 s} + \frac{1}{2\alpha^2 s^2} \;,$$

Multiplying by $\frac{v^2}{n+1}$ we obtain

$$\left(1+ \frac{v^2}{(n+1)(n+2)}\right)^{1/2} \geq \frac{1}{s} + \frac{2n+1}{2n+2}\;\frac{v^2}{\alpha^2 s} + \frac{1}{2(n+1)}\cdot\frac{v^2}{\alpha^2 s^2} \;,$$

$$\geq \frac{1 + \frac{v^2}{\alpha^2} - \frac{1}{2(n+1)}\,\frac{v^2}{\alpha^2}}{s} \;\; \frac{1}{2(n+1)}\;\frac{v^2}{\alpha^2 s^2} \;,$$

$$\geq s + \frac{v^2}{2(n+1)\alpha^2}\;\;\frac{(1 - s)}{s^2} \quad .$$

Squaring, we get

$$1 + \frac{v^2}{(n+1)(n+2)} \geq s^2 + \frac{v^4}{4(n+1)^2} \frac{1}{\alpha^4} \left[\frac{1-s}{s^2}\right]^2 + \frac{1}{s} \frac{v^2(1-s)}{(n+1)\alpha^2}$$

Transposing and using $s^2-1 = \frac{v^2}{\alpha^2}$,

$$\frac{v^2}{(n+1)(n+2)} - \frac{v^2}{\alpha^2} \geq \frac{v^2 (s^2-1)}{4(n+1)^2\alpha^2} \left(\frac{s-1}{s^2}\right)^2 + \frac{1}{s} \frac{v^2(1-s)}{(n+1)\alpha^2} .$$

Multiplying by $(n+1) \frac{\alpha^2}{v^2}$,

$$\frac{\alpha^2}{n+2} - n-1 \geq \frac{(s^2-1)}{4(n+1)} \left(\frac{s-1}{s^2}\right)^2 + \frac{1-s}{s} ,$$

and finally

$$\frac{\alpha^2}{n+2} - n \geq \frac{1}{s} + \frac{(s-1)^3(s+1)}{4(n+1)\, s^4} .$$

Let $F(s) = \frac{1}{s} + \frac{(s-1)^3(s+1)}{4(n+1)\, s^4}$. It is easily shown that $F(s)$ is a decreasing function in the range from 1 to ∞ and hence has a maximum at $s = 1$. Since $F(1)=1$, the inequality is valid if

$$\frac{\alpha^2}{n+2} - n \geq 1,$$

that is, if

$$\alpha^2 \geq (n+1)(n+2)$$

For $\alpha^2 = (n+1)(n+2)$, it is seen from (8.3) that $\Delta\tilde{X}$ is zero for $v = 0$ and negative for $v > 0$. This yields the

<u>Corollary</u> $x_{n+1} - x_n < \frac{1}{v^2} \cdot \frac{1}{\sqrt{1 + \frac{v^2}{(n+1)(n+2)}}}$ <u>for</u> $n \geq 0$ <u>and all</u> $v > 0$.

If we let

$$\tilde{Y}_n = y_n - \frac{n}{v^2} \frac{1}{\left(1 + \frac{v^2}{\alpha^2}\right)^{1/2}}$$

$$\Delta \tilde{Y} = \tilde{Y}_{n+1} - \tilde{Y}_n , \tag{8.5}$$

the pertinent formulas for $\Delta \tilde{Y}$ are:

$$\frac{d(\Delta \tilde{Y})}{dv} = v(\tilde{Y}_{n+1} + \tilde{Y}_n)\,\Delta \tilde{Y} + \frac{2n}{v}\,\Delta \tilde{Y} + \frac{2}{v}\,\frac{\tilde{Y}_{n+1}}{\sqrt{1 + \frac{v^2}{\alpha^2}}}$$

$$- \frac{2}{v}\,\Delta \tilde{Y} - \frac{2n+1}{v^3} + \frac{1}{v}\,\frac{1}{\alpha^2}\,\frac{1}{\left(1 + \frac{v^2}{\alpha^2}\right)^{3/2}} + \frac{2n+1}{v^3}\,\frac{1}{\left(1 + \frac{v^2}{\alpha^2}\right)} , \tag{8.6}$$

$$\Delta \tilde{Y} = \frac{1}{2}\left[\frac{1}{\alpha^2} - \frac{1}{n(n-1)}\right] + \frac{v^2}{8}\left[\frac{3n-2}{n^2(n-1)^2(n-2)} - \frac{3}{\alpha^4}\right] + \ldots \tag{8.7}$$

and

$$\Delta \tilde{Y} = \frac{1}{v^3}\left[(n + \tfrac{1}{2}) - \alpha\right] - \frac{2n+1}{2}\,\frac{1}{v^4} + \ldots \text{ for large } v \tag{8.8}$$

We will prove the following:

<u>Lemma 10</u>. $\Delta \tilde{Y} < 0$ <u>if</u> $\alpha \geq (n + \frac{1}{2})$ <u>for all</u> $n \geq 0$ <u>and</u> $v \geq 0$.

For $v = 0$, $\Delta \tilde{Y}$ is negative if $\alpha^2 > n(n-1)$ which is implied by the hypothesis of the lemma. At infinity $\Delta \tilde{Y}$ is negative for $\alpha \geq (n + \frac{1}{2})$. Thus if $\Delta \tilde{Y}$ ever became zero or positive, the graph of $\Delta \tilde{Y}$ would have a non-positive slope some point at which $\Delta \tilde{Y} = 0$. We shall prove the lemma by establishing a contradiction to this. Thus for $\Delta \tilde{Y} = 0$, we shall show that $\frac{d(\Delta \tilde{Y})}{dv}$ is positive whenever $\alpha \geq (n + \frac{1}{2})$. This amounts to showing that

$$\tilde{Y}_{n+1} \equiv y_{n+1} - \frac{n+1}{v^2}\,\frac{1}{s} > \frac{2n+1}{2\alpha^2}\,\frac{1}{s} - \frac{1}{2\alpha^2 s^2} \tag{8.9}$$

where $s^2 = 1 + \frac{v^2}{\alpha^2}$, as we see from (8.6). Recall that

$$y_{n+1} > \frac{n+1}{v^2} \left(1 + \frac{v^2}{(n+1)^2}\right)^{1/2} \quad \text{for } v > 0 \tag{8.10}$$

[see corollary 1 to theorem 8, Eq. (7.13)]. Using (8.10) in (8.9), we see that the lemma will be established if we can show that

$$\frac{n+1}{v^2}\left(1 + \frac{v^2}{(n+1)^2}\right)^{1/2} \geq \frac{n+1}{v^2}\,\frac{1}{s} + \frac{2n+1}{2\alpha^2 s} - \frac{1}{2\alpha^2 s^2},$$

or

$$\left(1 + \frac{v^2}{(n+1)^2}\right)^{1/2} \geq \frac{1 + \frac{v^2}{\alpha^2} - \frac{1}{2(n+1)}\frac{v^2}{\alpha^2}}{s} - \frac{1}{2(n+1)}\,\frac{v^2}{\alpha^2}\,\frac{1}{s^2}.$$

$$\geq s - \frac{1}{2(n+1)}\,\frac{v^2}{\alpha^2}\left(\frac{1+s}{s^2}\right). \tag{8.11}$$

Squaring (8.11) we obtain

$$1 + \frac{v^2}{(n+1)^2} \geq 1 + \frac{v^2}{\alpha^2} + \frac{1}{4(n+1)^2}\,\frac{v^4}{\alpha^4}\left(\frac{1+s}{s^2}\right)^2 - \frac{1}{(n+1)}\,\frac{v^2}{\alpha^2}\,\frac{(1+s)}{s}.$$

From which we get by transposing and dividing by v^2,

$$\frac{1}{(n+1)^2} - \frac{1}{\alpha^2} + \frac{1}{(n+1)\alpha^2} \geq \frac{1}{4(n+1)^2}\,\frac{v^2}{\alpha^4}\left(\frac{1+s)}{s^2}\right) - \frac{1}{(n+1)\alpha^2 s}.$$

Multiplying through by $(n+1)\alpha^2$ we have

$$\frac{\alpha^2}{(n+1)} - n \geq \frac{1}{4(n+1)}\,\frac{v^2}{\alpha^2}\left(\frac{1+s}{s^2}\right)^2 - \frac{1}{s},$$

or

$$\frac{\alpha^2}{(n+1)} - n \geq \frac{s^2 - 1}{4(n+1)}\left(\frac{1+s}{s^2}\right)^2 - \frac{1}{s}.$$

Let

$F(s) = \frac{(s-1)(s+1)^3}{4(n+1)\,s^4} - \frac{1}{s}$, then, since $F(s)$ is an increasing function, it has its maximum in the range from 1 to ∞ at $s = \infty$. Since

$F(\infty) = \frac{1}{4(n+1)}$, the inequality is valid if

$$\frac{\alpha^2}{(n+1)} \geq n + \frac{1}{4(n+1)} = \frac{(2n+1)^2}{4(n+1)} ,$$

that is for

$$\alpha^2 \geq (n + \tfrac{1}{2})^2 \tag{8.12}$$

Thus, in the range indicated by Eq. (8.12), $\frac{d(\Delta\widetilde{Y})}{dv}$ is positive for $\Delta\widetilde{Y} = 0$ and $v > 0$. The assumption that $\Delta\widetilde{Y}$ vanishes is therefore contrary to the behavior of $\Delta\widetilde{Y}$ at infinity so that the lemma is proved.

<u>Corollary</u> $\quad y_{n+1} - y_n < \frac{1}{v^2} \cdot \frac{1}{\sqrt{1 + \frac{v^2}{(n+\frac{1}{2})^2}}}$

<u>for all</u> $n \geq 0$ <u>and</u> $v \geq 0$.

We shall next sketch the proof of two lemmas which establish lower bounds on the functions $x_{n+1} - x_n$ and $y_{n+1} - y_n$. Although these lemmas will not be needed in the study of the W functions, they are of some interest in themselves.

<u>Lemma 11.</u> $\Delta\widetilde{X} > 0$ <u>if</u> $\alpha^2 \leq n(n+1)$ <u>for</u> $n \geq 0$ <u>and</u> $v \geq 0$.

Now $\Delta\widetilde{X}(0) > 0$ if $\alpha^2 < (n+1)(n+2)$, and $\Delta\widetilde{X}(\infty) > 0$ if $\alpha^2 < (n + \frac{1}{2})^2$, both of these inequalities are implied by $\alpha^2 \leq n(n+1)$. We shall show that $\frac{d(\Delta\widetilde{X})}{dv}$ is positive shenever $\Delta\widetilde{X}$ is equal to zero and $\alpha^2 \leq n(n+1)$. From (8.2) we see that this is equivalent to showing that

$$x_{n+1} < \frac{n+1}{v^2 s} + \frac{2n+1}{2\alpha^2 s} + \frac{1}{2\alpha^2 s^2} .$$

Using $x_{n+1} < \frac{n+1}{v^2}\left(1 + \frac{v^2}{(n+1)^2}\right)^{1/2}$, from the corollary to theorem 6, it is sufficient to show that

$$\left\{1 + \frac{v^2}{(n+1)^2}\right\}^{1/2} \leq \frac{1}{s} + \frac{2n+1}{2(n+1)} \frac{v^2}{\alpha^2 s} + \frac{1}{2(n+1)} \frac{v^2}{\alpha^2 s^2}$$

$$\leq s + \frac{1}{(2n+2)} \frac{v^2}{\alpha^2} \left(\frac{1-s}{s^2}\right)$$

Squaring and simplifying leads to

$$\frac{\alpha^2}{n+1} - n \leq \frac{1}{s} + \frac{1-s^2}{4(n+1)} \left(\frac{1-s}{s^2}\right)^2$$

The function on the right-hand side has its minimum at ∞ since it is a decreasing function. This minimum is $\frac{1}{4(n+1)}$. Hence $\Delta\widetilde{X} > 0$ if

$$\frac{\alpha^2}{n+1} - n < - \frac{1}{4(n+1)} ,$$

or

$$\alpha^2 < n^2 + n-1 < n(n+1)$$

Using lemmas 9 and 11 we obtain the following corollary:

Corollary. $$\frac{1}{v^2} \frac{1}{\left(1 + \frac{v^2}{n(n+1)}\right)^{1/2}} < x_{n+1} - x_n < \frac{1}{v^2} \frac{1}{\left(1 + \frac{v^2}{(n+1)(n+2)}\right)^{1/2}}$$

for $n \geq 0$ and $v > 0$.

Lemma 12. $\Delta\widetilde{Y} > 0$ if $\alpha^2 \leq n(n-1)$ for $n > 1$ and $v > 0$.

Eq. (8.8) shows that $\Delta\widetilde{Y}$ is positive for large v if $\alpha^2 \leq n(n-1)$. The present lemma will be established by showing that $\Delta\widetilde{Y} = 0$ implies that $\frac{d(\Delta\widetilde{Y})}{dv}$ is negative for the range of α indicated in the hypothesis, and therefore if $\Delta\widetilde{Y} = 0$ for any value of $v > 0$ it must thereafter remain negative, contrary to the above fact. That is, it must be shown that

$$y_{n+1} - \frac{n+1}{v^2 s} \equiv \widetilde{Y}_{n+1} < \frac{2n+1}{2v^2} s - \frac{2n+1}{2v^2 s} - \frac{1}{2\alpha^2 s^2}$$

where $s^2 = 1 + \frac{v^2}{\alpha^2}$. Or since

$$y_{n+1} < \frac{n+1}{v^2} \left(1 + \frac{v^2}{n(n+1)}\right)^{1/2} \quad \text{for } v > 0$$

by corollary 1 of theorem 8, it will be sufficient to show that

$$\frac{n+1}{v^2} \left(1 + \frac{v^2}{n(n+1)}\right)^{1/2} \leq \frac{n+1}{v^2 s} + \frac{2n+1}{2\alpha^2 s} - \frac{1}{2\alpha^2 s^2} ,$$

or

$$\left(1 + \frac{v^2}{n(n+1)}\right)^{1/2} \leq \frac{1 + \frac{v^2}{\alpha^2} - \frac{1}{2(n+1)}\frac{v^2}{\alpha^2}}{s} - \frac{1}{2(n+1)}\frac{v^2}{\alpha^2 s^2}$$

$$\leq s - \frac{1}{2(n+1)}\frac{v^2}{\alpha^2}\left(\frac{1+s}{s^2}\right)$$

Squaring and simplifying we find

$$\frac{\alpha^2}{n} - n \leq \frac{v^2}{4(n+1)\alpha^2}\left(\frac{1+s}{s^2}\right)^2 - \frac{1}{s}$$

$$\leq \frac{s^2-1}{4(n+1)}\frac{(s+1)^2}{s^4} - \frac{1}{s}$$

The function, $\frac{(s^2-1)(s+1)^2}{4(n+1)\,s^4} - \frac{1}{s}$, has a minimum in the range $(1, \infty)$ equal to -1 at s = 1, since it is an increasing function. Hence the inequality is valid if

$$\frac{\alpha^2}{n} - n \leq -1$$

or

$$\alpha^2 \leq n(n-1)$$

Corollary: $\frac{1}{v^2}\frac{1}{\left(1 + \frac{v^2}{n(n-1)}\right)^{1/2}} < y_{n+1} - y_n < \frac{1}{v^2}\frac{1}{\left(1 + \frac{v^2}{(n+\frac{1}{2})^2}\right)^{1/2}}$

for $n > 1$ and $v > 0$.

We conclude section 8 by finding bounds on the functions x_o, y_o and y_1.

Lemma 13. $a\left(1 + \frac{v^2}{\alpha^2}\right)^{-1/2} < x_o < \frac{1}{2}\left(1 + \frac{v^2}{4}\right)^{-1/2}$

for all $v > 0$, where $a \leq \frac{1}{2}$, $a^2\alpha^2 < 1 - \frac{a^2}{2}$.

In particular this inequality is true for $a = \frac{1}{2}$, $\alpha < \sqrt{\frac{1}{2}}$ and for $\alpha = \frac{1}{2}$, $a < \frac{\sqrt{2}}{3}$.

Define

$$\tilde{X}_o(v) \equiv \tilde{X}_o = x_o - a\left(1 + \frac{v^2}{\alpha^2}\right)^{-1/2}$$

The series expansion, the asymptotic formula and the differential equation for $\tilde{X}_o$ are:

$$\tilde{X}_o = \left(\frac{1}{2} - a\right) - \left(\frac{1}{16} - \frac{a}{2\alpha^2}\right) v^2 + \left(\frac{1}{12} - \frac{3a}{\alpha^4}\right) \frac{v^4}{8} + \ldots$$

$$\tilde{X}_o = (1 - a\alpha)\frac{1}{v} - \frac{1}{2v^2} + \ldots$$

$$\frac{d\tilde{X}_o}{dv} = - \tilde{X}_o^2 v - 2\left\{\frac{1}{v} + \frac{av}{\left(1 + \frac{v^2}{\alpha^2}\right)^{1/2}}\right\}\tilde{X}_o \tag{8.13}$$

$$+ \frac{1 + (1 - \alpha^2 a^2)\frac{v^2}{\alpha^2}}{v\left(1 + \frac{v^2}{\alpha^2}\right)} - \frac{a\left(2 + \frac{v^2}{\alpha^2}\right)}{v\left(1 + \frac{v^2}{\alpha^2}\right)^{3/2}} .$$

For $a = \frac{1}{2}$, $\alpha = 2$, $\tilde{X}_o$ is negative for small v; thus if $\tilde{X}_o$ vanishes for $v > 0$ its graph cuts the v-axis with non-negative slope. We shall show that $\frac{d\tilde{X}_o}{dv}$ is negative for any $v > 0$ for which $\tilde{X}$ vanishes, and hence prove the right-hand side the inequality, namely, $x_o < \frac{1}{2}\left(1 + \frac{v^2}{4}\right)^{-1/2}$. We must show (see Eq.8.13)

$$1 < \frac{\frac{1}{2}\left(2 + \frac{v^2}{4}\right)}{\left(1 + \frac{v^2}{4}\right)^{1/2}} .$$

Squaring we see that this is true since

$$1 < \frac{1 + \frac{v^2}{4} + \frac{v^4}{64}}{1 + \frac{v^2}{4}} = 1 + \frac{\frac{v^4}{64}}{1 + \frac{v^2}{4}} ,$$

For $a \leq \frac{1}{2}$ and $a\alpha < 1$, $\widetilde{X}_0$ is positive both near the origin and at infinity. It will be shown that $\widetilde{X}_0 = 0$ implies that $\frac{d\widetilde{X}_0}{dv}$ is positive. This will prove the remaining part of the lemma, namely $a(1 + \frac{v^2}{\alpha^2})^{-1/2} < x_0$.

Letting $\frac{v^2}{\alpha^2} = t$, $1 - \alpha^2 a^2 = A$, in (8.13), it must be shown that

$$1 + At > \frac{a(2 + t)}{(1 + t)^{1/2}} \quad \text{for } t > 0.$$

Squaring we must show that

$$1 + 2At + A^2 t^2 > 4a^2 + a^2 t \cdot \frac{t}{1+t} \quad ,$$

which is true if

$$1 + 2At > 4a^2 + a^2 t.$$

The last inequality holds if both

$$a^2 \leq \frac{1}{4} \text{ (that is } a \leq \frac{1}{2}\text{) and } 2A > a^2$$

that is if

$$2 - 2\alpha^2 a^2 > a^2 \text{ or } \alpha^2 a^2 < 1 - \frac{a^2}{2} \quad .$$

both of which are true by hypothesis.

<u>Lemma 14</u>. $y_0 > \frac{1}{2} (1 + \frac{v^2}{4})^{-1/2} \quad , \quad v > 0.$

Let
$$\widetilde{Y}_0 = y_0 - a\,(1 + \frac{v^2}{\alpha^2})^{-1/2}$$

The behavior of $\widetilde{Y}_0$ at the origin is seen from

$$\widetilde{Y}_0 = - \frac{1}{v^2 \log \frac{v}{2}} - a\,(1 + \frac{v^2}{\alpha^2})^{-1/2}$$

The asymptotic expansion and differential equation for $\widetilde{Y}_0$ are:

$$\widetilde{Y}_0 = (1 - a\alpha)\,\frac{1}{v} + \frac{1}{2v^2} + \ldots$$

$$\frac{d\tilde{Y}_o}{dv} = \tilde{Y}_o^2 v + 2 \left[av\left(1 + \frac{v^2}{\alpha^2}\right)^{-1/2} - \frac{1}{v} \right] \tilde{Y}_o + \frac{a^2 v}{\left(1 + \frac{v^2}{\alpha^2}\right)}$$

$$- \frac{2a}{v\left(1 + \frac{v^2}{\alpha^2}\right)^{1/2}} + \frac{av}{\alpha^2\left(1 + \frac{v^2}{\alpha^2}\right)^{3/2}} - \frac{1}{v} \quad .. \quad (8.14)$$

For $\alpha = 2$, $a = \frac{1}{2}$ $\tilde{Y}_o$ is positive for large v. In order that $\tilde{Y}_o$ may vanish it is necessary that $\frac{d\tilde{X}}{dv}$ be non-negative at some point at which $\tilde{Y}_o = 0$. We shall show that this is not so. Equation (8.14) (with $\check{Y}_o = 0$) becomes

$$\frac{d\tilde{Y}_o}{dv} = \frac{-1-(1-\alpha^2 a^2)\frac{v^2}{\alpha^2}}{v\left(1 + \frac{v^2}{\alpha^2}\right)} - \frac{a\left(2 + \frac{v^2}{\alpha^2}\right)}{v\left(1 + \frac{v^2}{\alpha^2}\right)^{3/2}}$$

which is clearly negative. The lemma is therefore proved.

By lemmas 13 and 14, we see that

$$y_o - x_o > \frac{1}{2}\left(1 + \frac{v^2}{4}\right)^{-1/2} - \frac{1}{2}\left(1 + \frac{v^2}{4}\right)^{-1/2} = 0,$$

which establishes the

<u>Corollary.</u> $y_o > x_o$ <u>for all</u> $v > 0$.

We conclude section 8 by proving the following two lemmas:

<u>Lemma 15.</u> $y_1 > \frac{1}{v^2} - \log \frac{\gamma v}{2}$ <u>for all</u> $v > 0$.

$[\gamma = 1.781 \text{ , } \log \gamma = .57772..]$.

<u>Lemma 16.</u> $y_1 < \frac{1}{v^2} - \log \frac{\gamma v}{2} + C(v_1)$ <u>for</u> $v < v_1$.

[$C(v_1)$ is a positive constant such that $y_1 = \frac{1}{v^2} - \log \frac{\gamma v_1}{2} + C(v_1)$] .

Lemma 15 is proved by letting

$$\tilde{Y}_1 = y_1 - \frac{1}{v^2} + \log \frac{\gamma}{2} v .$$

$\tilde{Y}_1$ then satisfies the differential equation

$$\frac{d\tilde{Y}_1}{dv} = v\,(\tilde{Y}_1 - \log \frac{\gamma v}{2})^2 \geq 0.$$

Since $\tilde{Y}_1(o) = 0$ and since $\tilde{Y}_1$ does not equal $\log \frac{\gamma v}{2}$ in any interval, it follows that $\tilde{Y}_1(v)$ is positive for all $v > 0$.

Lemma 16 follows from the observation that the function $y_1 - \frac{1}{v^2} + \log \frac{\gamma}{2} v - C$ is a monotonic increasing function of v which vanishes at $v = v_1$.

9. The Functions $W_{1,n}$ and $W_{2,n}$.

In section 9, we shall use the results of sections 5,6,7, and 8 to investigate the function $W_{1,n}$ and $W_{2,n}$ given by

$$\begin{aligned} W_{1,n}(v) &\equiv W_{1,n} = \frac{n}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}} + v_n , \\ W_{2,n}(v) &\equiv W_{2,n} = -\frac{n}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}} + v_n , \end{aligned} \qquad (4.2)$$

where

$$v_n(v) \equiv v_n = \sqrt{-\frac{1}{v^2}\frac{I_n'(v)\,K_n'(v)}{I_n(v)\,K_n(v)}} = \sqrt{x_n\,y_n} \qquad (4.4)$$

It will be shown that the functions $W_{1,n}$ and $W_{2,n}$ are ordered with respect to the index n, that is

$$\begin{aligned} W_{1,n+1} &> W_{1,n} \\ W_{2,n+1} &< W_{2,n} \end{aligned} \qquad (9.1)$$

The function $W_{1,n}$ will be shown to be a positive monotonic decreasing function of v. The section concludes with a partial description of $W_{2,n}(v)$ as a function of v for all values of the parameters n and α.

We shall follow the procedure of the preceding sections and base the proofs of the principal theorems on a series of lemmas.

<u>Lemma 17</u>. <u>The function</u> $w_n(v) = w_n = \sqrt{x_n y_n}$ <u>is a positive monotonic decreasing function of</u> v, <u>for</u> $v > 0$ <u>and all</u> $n \geq 0$.

Since x_n and y_n are both positive by lemmas 1 and 3 of section 5, it follows that w_n is real, and it is the positive square root of $x_n y_n$ by definition. Moreover

$$w_n' = \frac{x_n' y_n + y_n' x_n}{2\sqrt{x_n y_n}} .$$

Again x_n' and y_n' are negative (lemmas 1 and 3), hence w_n' is negative and the lemma is proved.

The next lemma concerns the ordering of the function w_n with respect to n.

<u>Lemma 18</u>. $w_{n+1} > w_n$ <u>for</u> $v \geq 0$ <u>and all</u> $n \geq 0$.

<u>We have</u>

$$w_{n+1} - w_n = \frac{x_{n+1} y_{n+1} - x_n y_n}{w_{n+1} + w_n}$$

$$= \frac{y_{n+1}(x_{n+1} - x_n) + x_n(y_{n+1} - y_n)}{w_{n+1} + w_n} .$$

Hence, by lemmas 1,2,3,4 and 17, it follows that

$$w_{n+1} - w_n > 0$$

for $v \geq 0$.

We are now in a position to prove

<u>Theorem 9</u>. $W_{1,n+1} > W_{1,n}$ <u>for</u> $v \geq$ <u>and all</u> $n \geq 0$.

By equation (4.2) we see that

$$W_{1,n+1} - W_{1,n} = \frac{1}{v^2}\sqrt{1 + \frac{v^2}{2}} + w_{n+1} - w_n .$$

Hence, $W_{1,n+1} - W_{1,n}$ is the sum of two positive terms and is itself positive.

The functions $\frac{n}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}}$ and w_n both have singularities at the origin and both approach the v-axis from above as v approaches infinity. Thus $W_{1,n}$ is initially plus infinity and approaches the v-axis from above. To complete the description of $W_{1,n}$ we prove

<u>Theorem 10.</u> $W_{1,n}$ <u>is a positive monotonic decreasing function of</u> v <u>for</u> $v > 0$ <u>and</u> $n \geq 0$.

<u>Proof:</u> $W_{1,n}$ is the sum of two positive functions both of which are monotonic decreasing.

Fig. 13 indicates graphically the content of theorems 9 and 10.

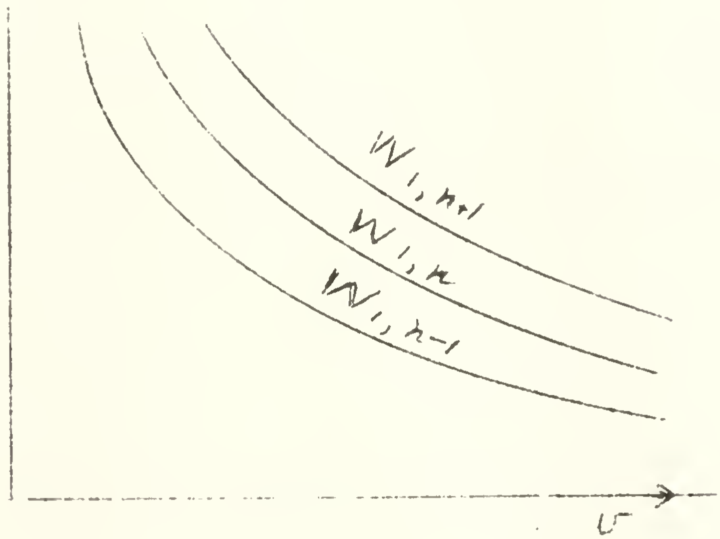


FIG. 13

We now come to the function $W_{2,n}$. As might be expected from sections 6 and 7, the proof of the ordering theorem for the functions $W_{2,n}$ is more difficult than that for $W_{1,n}$. It depends on the inequality $w_{n+1} - w_n < \frac{1}{v^2}$, to the proof of which we devote the next two lemmas.

Lemma 19. $w_{n+1} - w_n < \frac{1}{v^2}$ for $v > 0$ and $n \geq 1$.

Using the identity

$$w_{n+1} - w_n = \frac{w_{n+1}^2 - w_n^2}{w_{n+1} + w_n} ,$$

it must be shown that

$$\frac{x_{n+1}\, y_{n+1} - x_n\, y_n}{\sqrt{x_{n+1} y_{n+1}} + \sqrt{x_n\, y_n}} < \frac{1}{v^2} ,$$

or that

$$y_{n+1}\,(x_{n+1} - x_n) + x_n\,(y_{n+1} - y_n) < \frac{1}{v^2}\ \sqrt{x_{n+1} y_{n+1}} + \sqrt{x_n\, y_n} \qquad (9.2)$$

It is now apparent why bounds on the functions $x_n, y_n,$ $x_{n+1} - x_n$ and $y_{n+1} - y_n$ were found in sections 6, 7 and 8.

We break up the inequality (9.2) into two inequalities which imply (9.2):

$$x_n(y_{n+1} - y_n) < \frac{1}{v^2}\sqrt{x_n\, y_n} \qquad (9.3)$$

$$y_{n+1}\,(x_{n+1} - y_n) < \frac{1}{v^2}\,\sqrt{x_{n+1}\, y_{n+1}} \quad . \qquad (9.4)$$

Here we replace $(y_{n+1} - y_n)$ and $(x_{n+1} - x_n)$ by bounds established in corollaries to lemmas 9 and 10, and y_n and x_{n+1} by bounds established in the corollaries to theorems 6 and 8. The inequality (9.3) is then valid if

$$x_n \frac{1}{v^2\,(1 + \frac{v^2}{(n+\frac{1}{2})^2})^{1/2}} \leq \frac{\sqrt{x_n}}{v^2}\,\frac{\sqrt{n}}{v}\,(1 + \frac{v^2}{n^2})^{1/4} ,$$

that is if

$$\sqrt{x_n} \leq \frac{\sqrt{n}}{v}\,(1 + \frac{v^2}{n^2})^{1/4}\,(1 + \frac{v^2}{(n + \frac{1}{2})^2})^{1/2} \quad .$$

Squaring we obtain

$$x_n \leq \frac{n}{v^2}\left(1 + \frac{v^2}{n^2}\right)^{1/2}\left(1 + \frac{v^2}{(n+\frac{1}{2})^2}\right) . \tag{9.5}$$

Now by the corollary to theorem 6,

$$x_n < \frac{n}{v^2}\left(1 + \frac{v^2}{n^2}\right)^{1/2} ,$$

which clearly implies (9.5) .

In like manner (9.4) reduces to

$$\sqrt{y_{n+1}} \leq \frac{\sqrt{n+1}}{v}\left(1 + \frac{v^2}{(n+1)(n+2)}\right)^{3/4}$$

Squaring we get

$$y_{n+1} \leq \frac{n+1}{v^2}\left(1 + \frac{v^2}{(n+1)(n+2)}\right)^{3/2}$$

But (Corollary to theorem 8)

$$y_{n+1} < \frac{n+1}{v^2}\sqrt{1 + \frac{v^2}{n(n+1)}} ,$$

so that we want

$$\frac{n+1}{v^2}\sqrt{1 + \frac{v^2}{n(n+1)}} \leq \frac{n+1}{v^2}\left(1 + \frac{v^2}{(n+1)(n+2)}\right)^{3/2} ,$$

or

$$1 + \frac{v^2}{n(n+1)} \leq 1 + \frac{3v^2}{(n+1)(n+2)} + \frac{3v^4}{(n+1)^2(n+2)^2} + \frac{v^6}{(n+1)^3(n+2)^3}$$

This will follow from

$$\frac{1}{n(n+1)} \leq \frac{3}{(n+1)(n+2)}$$

or

$$n+2 \leq 3n$$

$$1 \leq n$$

which proves (9.4) and the lemma.

The bounds established in Section 8 which have been used to prove the above inequalities are not sharp enough to deal with the case $n = 0$ for small v. Hence there remains the case $n = 0$ which is treated in lemma 20.

Lemma 20. $w_1 - w_o < \frac{1}{v^2}$, $v > 0$.

As in lemma 19, it is to be shown that

$$\sqrt{x_1 y_1} - \sqrt{x_o y_o} < \frac{1}{v^2} \quad \text{or}$$

$$y_1(x_1 - x_o) + x_o(y_1 - y_o) < \frac{1}{v^2}\left\{\sqrt{x_1 y_1} + \sqrt{x_o y_o}\right\} . \tag{9.6}$$

The proof of (9.6) is divided into two parts: a) $v \geq 0.8$ and b) $v < 0.8$.

a). First we show that

$$x_o(y_1 - y_o) < \frac{1}{v^2} \sqrt{x_o y_o} . \tag{9.7}$$

Using the corollary to lemma 10 we have

$$x_o(y_1 - y_o) < \frac{1}{v^2} \frac{1}{\sqrt{1 + 4v^2}}$$

and from lemma 14

$$\frac{1}{v^2} \sqrt{x_o y_o} > \frac{1}{v^2} \sqrt{x_o} \frac{1}{v^2} \frac{1}{\sqrt[4]{1 + \frac{v^2}{4}}}$$

Hence (9.7) will follow if we can show

$$x_o \cdot \frac{1}{v^2} \frac{1}{\sqrt{1 + 4v^2}} \leq \frac{1}{v^2} \sqrt{x_o} \frac{1}{\sqrt{2}} \frac{1}{\sqrt[4]{1 + \frac{v^2}{4}}}$$

OR

$$\sqrt{x_o} \leq \frac{1}{\sqrt{2}} \frac{\sqrt{1 + 4v^2}}{\sqrt[4]{1 + \frac{v^2}{4}}} ,$$

that is

$$x_o \leq \frac{1}{2} \quad \frac{1 + 4v^2}{\sqrt{1 + \frac{v^2}{4}}} \tag{9.8}$$

However (lemma 13)

$$x_o < \frac{1}{2}(1 + \frac{v^2}{4})^{-1/2} \quad .$$

Thus (9.8) is true and hence (9.7) is true for all $v > 0$.

We must now show that

$$y_1 (x_1 - x_o) < \frac{1}{v^2} \sqrt{x_1 y_1} \quad ,$$

which, by the corollaries to lemma 9 and theorem 6 will follow from

$$\sqrt{y_1} \quad \frac{1}{v^2} \cdot \frac{1}{\sqrt{1 + \frac{v^2}{2}}} < \frac{1}{v^2} \cdot \frac{1}{v} \sqrt[4]{1 + \frac{v^2}{2}}$$

or

$$y_1 < \frac{1}{v^2} (1 + \frac{v^2}{2})^{3/2} \quad . \tag{9.9}$$

Eq. (9.9) is certainly false for v small, as seen from Eq. (5.42). It will be shown to be true for $v \geq 0.8$. We begin by showing that

$$y_1 < \frac{1}{v^2} \quad (1 + 2.03\, v^2)^{1/2} \quad \text{for } v \geq 0.8. \tag{9.10}$$

The function $y_1 - \frac{1}{v^2}(1 + 2.03\, v^2)^{1/2}$ is a $Y_{2,1}$ function with $\alpha < 1$. Thus (see theorem 8) if $Y_{2,1}$ is negative for any value of v (say v = .8) it will be negative thereafter.

But

$$y_1(v) = \frac{1 + v \dfrac{K_o(v)}{K_1(v)}}{v^2} \quad ,$$

so that

$$y_1(.8) = \frac{1 + .8 \dfrac{1.2582}{1.9179}}{(0.8)^2} = \frac{1.5247}{(0.8)^2} \quad ;$$

whereas

$$\frac{1}{(0.8)^2} \quad (1 + 2.03(0.8)^2)^{1/2} = \frac{1.526}{(0.8)^2} \quad .$$

There remains to show that

$$\frac{1}{v^2} \quad (1 + 2.03 \, v^2)^{1/2} \leq \frac{1}{v^2} \quad (1 + \frac{v^2}{2})^{3/2}$$

for $v \geq 0.8$. Squaring we must show that

$$f(v) = \frac{v^4}{8} + \frac{3}{4} v^2 - .53 \geq 0 \quad \text{for } v \geq 0.8 \; .$$

$f(v)$ is negative for $v = 0$; $f(v) = 0$ has a positive root and is positive thereafter; so that if $f(.8) > 0$, it follows that $f(v) > 0$ for $v \geq .8$. However, $f(.8) = .0012 > 0$. This proves part (a) of the lemma.

(b). We shall show for $v < 0.8$ that

$$\sqrt{x_1 y_1} < \frac{1}{v^2} + \sqrt{x_o y_o} \quad ,$$

or equivalently that

$$x_1 y_1 < \frac{1}{v^4} + \frac{2\sqrt{x_o y_o}}{v^2} + x_o y_o \quad .$$

To accomplish this we get appropriate bounds on the four functions x_1, y_1, x_o and y_o.

An equivalent form to lemma 16 may be written as

$$y_1 < \frac{1}{v^2} - \log kv \quad , \qquad \text{for } v < v_1 \tag{9.11}$$

where k depends on v_1 . We choose k so that the function $y_1 - \frac{1}{v^2} + \log kv = 0$ at $v = 0.8$; that is

$$1 + v \; \frac{K_o(v)}{K_1(v)} - 1 + v^2 \log kv = 0 \quad \text{for } v = .8.$$

Making the calculation we get $\qquad k = \dfrac{1}{1.81632}$.

Again

$$y_1 = \frac{1}{v^2}\left[1 + v\,\frac{K_o}{K_1}\right] = \frac{1}{v^2} + v\,y_o$$

so that $y_o = \dfrac{1}{v^2 y_1 - 1}$. From (9.11) we have

$$v^2\,y_1 - 1 < 1 - v^2 \log kv - 1 = -v^2 \log kv$$

which leads to

$$y_o = \frac{1}{v^2 y_1 - 1} > \frac{1}{-v^2 \log kv} \quad . \tag{9.12}$$

The function $x_1 - \frac{1}{v^2}\sqrt{1 + \frac{4}{7}v^2}$ is an $X_{2,1}$ function with

$n = 1 < \frac{\sqrt{7}}{2} = \alpha < \sqrt[4]{12} = \sqrt[4]{n(n+1)^2(n+2)}$ so that if $X_{2,1}$ is negative for, say, $v = 1$, then it is negative for all $v \leq 1$ It turns out that $X_{2,1}(1) = -.01$.

Hence

$$x_1 < \frac{1}{v^2}\sqrt{1 + \frac{4}{7}\,v^2} < \frac{1}{v^2}\sqrt{1 + \frac{4}{7}v^2 + \frac{4}{49}\,v^4} = \frac{1}{v^2}(1 + \frac{2}{7}\,v^2) \text{ for } \tag{9.13}$$

$v \leq 0.8$

Finally x_o is a positive monotonic decreasing function which means that

$$x_o(v) \geq x_o(.8) \quad \text{for all } v < 0.8 \; .$$

However, $x_o(.8) = \frac{1}{v}\,\frac{I_1}{I_o} = .4637 > .36$

so that

$$\sqrt{x_o(v)} > .6 \quad \text{for } v < 0.8 \; . \tag{9.14}$$

Using (9.11),(9.12),(9.13) and (9.14) in

$$x_1 y_1 < \frac{1}{v^4} + \frac{2}{v^2}\sqrt{x_o y_o} + x_o y_o \quad ,$$

we get

$$\frac{1}{v^2}\left[1 + \frac{2}{7}\,v^2\right]\left[\frac{1}{v^2} - \log kv\right] \leq \frac{1}{v^4} + \frac{1.2}{v^2}\sqrt{\frac{-1}{v^2 \log kv}} \; , \tag{9.15}$$

where the last term on the right-hand side has been neglected. Inequality (9.15) simplifies to

$$\frac{2}{7}\ \frac{1}{v^2} - \frac{2}{7}\ \log kv - \frac{1}{v^2}\ \log kv < \frac{1.2}{v^2}\sqrt{\frac{-1}{v^2 \log kv}}\ ,$$

or

$$\frac{2}{7}\sqrt{-\ v^2 \log kv} - \frac{2}{7}\ v^2 \log kv \sqrt{-v^2 \log kv} - \log kv \sqrt{-v^2 \log kv} \leq 1.2 \qquad (9.16)$$

The functions $-v^2 \log kv$ and $-\log kv \sqrt{-v^2 \log kv}$ have maxima for the values $kv = e^{-1/2}$, $kv = e^{-3/2}$, respectively. Replacing the left-hand terms in (9.16) by their maxima, we get

$$\frac{2}{7}\sqrt{\frac{1}{2}\ \frac{e^{-1}}{k^2}} + \frac{2}{7}\left(\frac{1}{2}\ \frac{e^{-1}}{k^2}\right)^{3/2} + \frac{3}{2}\sqrt{\frac{3}{2}\ \frac{e^{-3}}{k^2}} \leq 1.2\ ,$$

or

$$\frac{2}{7}\sqrt{\frac{1}{2}\ \frac{e^{-1}}{k^2}}\left[1 + \frac{1}{2}\ \frac{e^{-1}}{k^2}\right] + \frac{3}{2}\sqrt{\frac{3}{2}\ \frac{e^{-3}}{k^2}} \leq 1.2\ .$$

Making the computation we get

$$.361 + .729 = 1.09 < 1.20$$

which proves the lemma.

Consider

$$W_{2,n+1} - W_{2,n} = -\ \frac{1}{v^2}\sqrt{1 + \frac{v^2}{\alpha^2}} + w_{n+1} - w_n$$

$$= \frac{1}{v^2}\ \ 1 - \sqrt{1 + \frac{v^2}{\alpha^2}} + w_{n+1} - w_n - \frac{1}{v^2}\ .$$

The function $W_{2,n+1} - W_{2,n}$ is therefore, by lemma 19, the sum of two negative terms, hence is itself negative. This proves the fundamental

Theorem 11. *$W_{2,n} > W_{2,n+1}$ for $v > 0$ and all $n \geq 0$.*

We have not been able to prove the analogues to theorem 6 and 8, that is a theorem which characterizes $W_{2,n}(v)$ as a function of v for all values of the parameters n and α. We do, however, state and prove two theorems which give a partial answer to the above problem.

The series and asymptotic formula for $W_{2,n}$ are:

$$W_{2,n} = \frac{1}{2}\left[\frac{n}{n^2-1} - \frac{n}{\alpha^2}\right] - n\,\frac{v^2}{8}\left[\frac{n^4+6n^2-4}{n^2(n^2-1)^2(n^2-4)} - \frac{1}{\alpha^4}\right] + \ldots \quad (9.17)$$

$$W_{2,n} = \frac{1}{v}\left[1 - \frac{n}{\alpha}\right] + \frac{1}{2v^3}\left[n^2 - n\alpha - \frac{1}{2}\right] + \ldots \quad (9.18)$$

<u>Lemma 21.</u> <u>$W_{2,n}$ is positive if $X_{2,n}$ and $Y_{2,n}$ are positive; $W_{2,n}$ is negative if $X_{2,n}$ and $Y_{2,n}$ are negative.</u>

For $X_{2,n} > 0$ and $Y_{2,n} > 0$, that is, $\sqrt{x_n} > \frac{\sqrt{n}}{v}\sqrt[4]{1+\frac{v^2}{\alpha^2}}$ and $\sqrt{y_n} > \frac{\sqrt{n}}{v^2}\sqrt[4]{1+\frac{v^2}{\alpha^2}}$, we have $\sqrt{x_n y_n} > \frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha}}$,

which implies that

$$W_{2,n} > 0.$$

In exactly the same way it follows that if

$$X_{2,n} < 0 \text{ and } Y_{2,n} < 0$$

then

$$W_{2,n} < 0.$$

Since, from the corollaries to theorems 6 and 8, $X_{2n} < 0$ for $\alpha \leq n$ and $Y_{2,n} \leq 0$ for $\alpha \leq \sqrt{n(n-1)}$, we obtain, as a corollary to this lemma that $W_{2,n} < 0$ if $\alpha \leq \sqrt{n(n-1)}$. Similarly, $W_{2,n} > 0$ if $\alpha \geq \sqrt{n(n+1)}$.

<u>Theorem 12.</u> $W_{2,n}$ <u>is a monotonic decreasing function of</u> v <u>if</u>

$\alpha > \sqrt[4]{n(n+1)^2(n+2)}$ and $v > 0$.

The hypothesis implies (see theorems 6 and 8) that both $X_{2,n}$ and $Y_{2,n}$ are monotonic decreasing. We have:

$$W_{2,n} = -\varphi_n(v,\alpha) + \sqrt{x_n y_n}$$

where

$$\varphi_n(v,\alpha) = \frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} \quad . \tag{4.10}$$

Differentiating, we get

$$W'_{2,n} = \frac{x_n y'_n + x'_n y_n}{2\sqrt{x_n y_n}} - \varphi'_n(v,\alpha)$$

$$= \frac{1}{2}\sqrt{\frac{x_n}{y_n}}\, Y'_{2,n} + \frac{1}{2}\sqrt{\frac{y_n}{x_n}}\, X'_{2,n} + \left(\frac{x_n + y_n}{2\sqrt{x_n y_n}} - 1\right)\varphi'_n(v,\alpha)$$

But $X'_{2,n}$, $Y'_{2,n}$, φ'_α are negative and $x_n + y_n > 2\sqrt{x_n y_n}$. Hence $W'_{2,n}$ is negative.

The last theorem we shall prove is the following:

<u>Theorem 13.</u> $W_{2,n}$ <u>is a monotonic increasing function of</u> v <u>if</u>

$\alpha \leq \sqrt[4]{n(n-1)^2(n-2)}$ <u>and</u> $v > 0$ <u>and</u> $n \geq 3$.

By the corollaries of theorems 6 and 8, we have for $v > 0$

$$\varphi_n(v, \sqrt{n(n+1)}) < x_n < \varphi_n(v,n); \qquad \varphi_n(v,n) < y_n < \varphi_n(v, \sqrt{n(n-1)}) \tag{9.19}$$

$$0 > y'_n > \varphi'_n(v, \sqrt[4]{n(n-1)^2(n-2)}), \quad 0 > x'_n > \varphi'_n(v,n)$$

From (9.19) it follows that

$$\frac{\varphi_n(v, \sqrt{n(n+1)}\ )}{\varphi_n(v, \sqrt{n(n-1)}\ )} < \frac{x_n}{y_n} < 1 . \tag{9.20}$$

We shall prove that

$$W'_{2,n} = -\varphi'_n(v, \alpha) + \frac{1}{2}\sqrt{\frac{x_n}{y_n}}\ y'_n + \frac{1}{2}\frac{x'_n}{\sqrt{\frac{x_n}{y_n}}} > 0 . \tag{9.21}$$

if $\alpha \leq \sqrt[4]{n(n-1)^2(n-2)}$ and $v > 0$.

Let $r = \sqrt{\frac{x_n}{y_n}}$, then (9.21) becomes

$$-2\varphi'_n(v, \alpha) > -ry'_n - \frac{1}{r}\ x'_n \quad ,$$

which by virtue of (9.19) will follow if we can show that

$$-2\ \varphi'_n(v, \alpha) > r\ (\ -\ \varphi'_n(v, \sqrt[4]{n(n-1)^2(n-2)}\big) + \frac{1}{r}\ (\ -\varphi'_n(v, n). \tag{9.22}$$

It is easy to show that for $\alpha \leq \sqrt[4]{n(n-1)^2(n-2)}$

$$-2\ \varphi'_n(v, \alpha) \geq -2\ \varphi'_n\ (v, \sqrt[4]{n(n-1)^2(n-2)}\ ) \quad ,$$

which with (9.22) yields the inequality that will be proved, namely,

$$(2-r)\ r\ (\ -\ \varphi'_n\ (v, \sqrt[4]{n(n-1)^2(n-2)}))> -\ \varphi'_n(v,n). \tag{9.23}$$

Inequality (9.23) must be proved for all r in the range (see 9.20) .

$$\sqrt{\frac{\varphi_n(v, \sqrt{n(n+1)}}{\varphi_n(v, \sqrt{n(n-1)}\ )}} < r < 1 . \tag{9.24}$$

The function $(2-r)\, r \left(-\varphi_n'\left(v, \sqrt[4]{n(n-1)^2(n-2)}\right) \right) + \varphi_n'(v,n)$ (see (9.23) considered as a function of r is a parabolic function having its maximum at $r = 1$. This function therefore has its minimum in the range specified in (9.24) for

$r = \sqrt{\dfrac{\varphi_n(v, \sqrt{n(n+1)})}{\varphi_n(v, \sqrt{n(n-1)})}}$. Hence if the inequality of (9.23) is proved for this value of r it must hold for all values of r in the range of (9.24)

For

$$r = \sqrt{\frac{\varphi_n(v, \sqrt{n(n+1)})}{\varphi_n(v, \sqrt{n(n-1)})}} = \sqrt[4]{\frac{1 + \dfrac{v^2}{n(n+1)}}{1 + \dfrac{v^2}{n(n-1)}}} , \qquad (9.25)$$

we normalize the parabolic function by the substitution $\sigma = 1 - r$. If in (9.23) we replace $\varphi_n'(v,n)$, $\varphi_n'\left(v, \sqrt[4]{n(n-1)^2(n-2)}\right)$ by their values and r by $1 - \sigma$, we have

$$(1 - \sigma^2) \frac{2 + \dfrac{v^2}{\sqrt{n(n-1)^2(n-2)}}}{\left(1 + \dfrac{v^2}{\sqrt{n(n-1)^2(n-2)}}\right)^{1/2}} > \frac{2 + \dfrac{v^2}{n^2}}{\sqrt{1 + \dfrac{v^2}{n^2}}} . \qquad (9.26)$$

The next part of the proof is concerned with obtaining an upper bound for σ. Now

$$\sigma = 1 - r = \frac{1 - r^4}{(1 + r)(1 + r^2)}$$

It is easy to find a lower bound for the denominator since r is a monotonic decreasing function of v with a minimum of $\sqrt[4]{\dfrac{n-1}{n+1}} \geq \sqrt[4]{\dfrac{1}{2}}$ for $n \geq 3$.

Hence $(1 + r)(1 + r^2) \geq (1.83)(1.71) > 3$. Combining this with Eq. (9.25) we get

$$\sigma < \frac{1-r^4}{3} = \frac{2v^2}{3n(n^2-1)} \cdot \frac{1}{1 + \frac{v^2}{n(n-1)}} \quad . \tag{9.27}$$

Finally we show that

$$\left\{ 1 + \frac{v^2}{n(n-1)} \right\}^2 > \left(1 + \frac{v^2}{n^2}\right) \left(1 + \frac{v^2}{2\sqrt{n(n-1)^2(n-2)}}\right) \text{ for } n \geq 3 \tag{9.28}$$

In fact, if we expand both sides we obtain

$$1 + \frac{2v^2}{n(n-1)} + \frac{v^4}{n^2(n-1)^2} > 1 + v^2 \left[\frac{1}{n^2} + \frac{1}{\sqrt{n(n-1)^2(n-2)}}\right] + \frac{v^4}{2n^2 \sqrt[4]{n(n-1)^2(n-2)}}$$

which will certainly be true if the coefficients of like powers are subject to like inequalities.

For v^2 we want

$$\frac{2}{n(n-1)} - \frac{1}{n^2} > \frac{1}{2\sqrt{n(n-1)^2(n-2)}} \quad ,$$

or

$$\frac{n+1}{n^2(n-1)} > \frac{1}{2(n-1)\sqrt{n(n-2)}} \quad ,$$

or after some simplification

$$3n^3 - 12n - 8 > 0 \, .$$

But

$$3n^3 - 12n - 8 = 37 + 69(n-3) + 27(n-3)^2 + 3(n-3)^3 > 0, \text{ for } n \geq 3.$$

For v^4, we want

$$\frac{1}{n^2(n-1)^2} > \frac{1}{2n^2\sqrt{n(n-1)^2(n-2)}}$$

which will be true if

$$3n^2 - 6n-1 > 0.$$

Again

$$3n^2 - 6n - 1 = 3(n-3)^2 + 12(n-3) + 8 > 0 \text{ for } n \geq 3.$$

Thus (9.28) is established.

Using (9.28) in (9.27) we get

$$\sigma^2 < \frac{4}{9} \frac{v^4}{n^2(n^2-1)^2} \cdot \frac{1}{1+\frac{v^2}{n^2}} \cdot \frac{1}{1+\frac{v^2}{2\sqrt{n(n-1)^2(n-2)}}} \cdot \tag{9.29}$$

We now proceed to prove inequality (9.26). Squaring, we get

$$(1-\sigma^2)^2 \left(1+\frac{v^2}{n^2}\right) \left(4 + \frac{4v^2}{\sqrt{n(n-1)^2(n-2)}} + \frac{v^4}{n(n-1)^2(n-2)}\right)$$

$$> \left(4 + \frac{4v^2}{n^2} + \frac{v^4}{n^4}\right) \left(1 + \frac{v^2}{\sqrt{n(n-1)^2(n-2)}}\right) . \tag{9.30}$$

But $(1-\sigma^2)^2 = 1 - 2\sigma^2 + \sigma^4 > 1 - 2\sigma^2$ since $0 < \sigma < 1 - r < 1$. So that, from (9.29),

$$(1-\sigma^2)^2 > 1 - 2\sigma^2 > 1 - \frac{8}{9} \frac{v^4}{n^2(n^2-1)^2} \cdot \frac{1}{1+\frac{v^2}{n^2}} \cdot \frac{1}{1+\frac{v^2}{2\sqrt{n(n-1)^2(n-2)}}}.$$

It follows that

$$(1-\sigma^2)^2 > 1 - \frac{v^4}{n^2(n^2-1)^2} \cdot \frac{1}{+\frac{v^2}{n^2}} \cdot \frac{1}{1+\frac{v^2}{2\sqrt{n(n-1)^2(n-2)}}} \tag{9.31}$$

Substituting the inequality (9.31) in (9.30) and expanding, we get

$$4 + \left[\frac{4}{n^2} + \frac{4}{\sqrt{n(n-1)^2(n-2)}}\right]v^2 + \left[\frac{1}{n(n-1)^2(n-2)} + \frac{4}{n^2\sqrt{n(n-1)^2(n-2)}}\right]v^4 + \frac{v^6}{n^3(n-1)^2(n-2)}$$

$$- \frac{v^4}{n^2(n^2-1)^2} \cdot \frac{1}{1+\frac{v^2}{n^2}} \cdot \frac{1}{1+\frac{v^2}{2\sqrt{n(n-1)^2(n-2)}}}\, 4\left(1+\frac{v^2}{n^2}\right)\left(2+\frac{v^2}{\sqrt{n(n-1)^2(n-2)}}\right)^2 \tag{9.32}$$

$$> 4 + \left[\frac{4}{n^2} + \frac{4}{\sqrt{n(n-1)^2(n-2)}}\right]v^2 + \left[\frac{1}{n^4} + \frac{4}{n^2\sqrt{n(n-1)^2(n-2)}}\right]v^4 + \frac{v^6}{n^4\sqrt{n(n-1)^2(n-2)}}$$

or

$$\frac{v^4}{n(n-1)^2(n-2)} + \frac{v^6}{n^3(n-1)^2(n-2)} - \frac{16v^4}{n^2(n^2-1)^2}\left(1+\frac{v^2}{2\sqrt{n(n-1)^2(n-2)}}\right)$$

$$> \frac{v^4}{n^4} + \frac{v^6}{n^4\sqrt{n(n-1)^2(n-2)}} .$$

We shall establish this last inequality by comparing the different powers of v. For v^4 we have

$$\frac{1}{n(n-1)^2(n-2)} > \frac{1}{n^4} + \frac{16}{n^2(n^2-1)^2} = \frac{n^4+14n^2+1}{n^4(n^2-1)^2}$$

or

$$\frac{1}{n-2} > \frac{n^4+14n^2+1}{n^3(n+1)^2} .$$

This follows if

$$4n^4 - 13n^3 + 28n^2 - n + 2 = 4(n-1)^4 + 3(n-1)^3 + 18(n-1)^2 + 32(n-1) + 20 > 0,$$

which is true for $n > 1$.

For v^6 we want

$$\frac{1}{n^3(n-1)^2(n-2)} > \frac{1}{n^4\sqrt{n(n-1)^2(n-2)}} + \frac{8}{n^2(n^2-1)^2\ \sqrt{n(n-1)^2(n-2)}}$$

or

$$\frac{1}{\sqrt{n(n-1)^2(n-2)}} > \frac{1}{n^2} + \frac{8}{(n^2-1)^2} = \frac{n^4+6n^2+1}{n^2(n^2-1)^2} ,$$

or

$$\frac{1}{\sqrt{n(n-2)}} > \frac{n^4+6n^2+1}{n^2(n+1)^2(n-1)}$$

Now

$$\frac{1}{n-1} = \frac{n^4+2n^3+n^2}{n^2(n+1)^2(n-1)} > \frac{n^4+6n^2+1}{n^2(n+1)^2(n-1)} ,$$

for $n \geq 3$, since

$$2n^3 - 5n^2 + 1 = 2(n-3)^3 + 13(n-3)^2 + 24(n-3) + 10 > 0 \quad \text{for } n \geq 3.$$

Hence (9.32) is established and the theorem is proved.

The bounds on α in theorems 12 and 13 were suggested by theorems 6 and 8. They are not the best possible bounds. In connection with theorem 13, let us remark that if $W_{2,n}(v,\alpha_1)$ is monotonic increasing, it follows that

$W_{2,n}(v,\alpha_1) > W_{2,n}(v,\alpha_2)$ and that $W_{2,n}(v,\alpha_2)$

is monotonic increasing for $\alpha_2 < \alpha_1$ (since $\phi_n'(v,\alpha_1) < \phi_n'(v,\alpha_2)$).

10. Conclusions.

To return to the non-attenuated modes of a helical wave guide, let us see what can now be said about the real solutions of

$$\frac{\delta}{\alpha} = -\frac{n}{v^2}\sqrt{1+\frac{v^2}{\alpha^2}} \pm \sqrt{-\frac{I_n'(v)\,K_n'(v)}{v^2\,I_n(v)\,K_n(v)}} . \qquad (2.6)$$

The left-hand side of this equation is a positive constant; the right-hand side can be written in terms of the functions $W_{1,n}$ and $W_{2,n}$. Equation (2.6) then becomes

$$\left.\begin{aligned} \frac{\delta}{\alpha} &= W_{2,n}(v) & n \geq 0 \\ &= W_{1,|n|}(v) & n \leq 0 \end{aligned}\right\} \mathrm{I}$$

$$\left.\begin{aligned} \frac{\delta}{\alpha} &= -W_{1,n}(v) & n \geq 0 \\ &= W_{2,|n|}(v) & n \leq 0 \end{aligned}\right\} \mathrm{II} \qquad (10.1)$$

Group I corresponds to (2.6) with a plus sign before the radical and group II to (2.6) with a minus sign before the radical

It follows from theorems 9 and 11 that the functions in group I are ordered. That is

$$-- > W_{1,|-2|} > W_{1,|-1|} > W_{1,0} = W_{2,0} > W_{2,1} > W_{2,2} > ---$$

for each $v \geq 0$. This is illustrated in Fig. 1. The functions in group II are clearly the negatives of the functions in group I and hence are ordered as

$$-- > -W_{2,|-2|} > -W_{2,|-1|} > -W_{2,0} = -W_{1,0} > -W_{1,1} > -W_{1,2} > ---$$

for each $v \geq 0$. By theorem 10, the $W_{1,n}$ functions are positive, start at infinity, are monotonic decreasing, and approach zero asymptotically as $v \to \infty$. Hence for group I there is precisely one solution for each $n \leq 0$; if v_n is the solution for the nth mode, then the solutions are ordered as

$$0 < v_0 < v_{-1} < v_{-2} < --- \qquad (10.2)$$

For group II there are no solutions for $n \geq 0$.

The solutions of Eq. (10.1) associated with the $W_{2,n}$ functions are more difficult to describe. For sufficiently large n (in practice, $n \geq 3$) we will have $\sqrt[4]{n(n-1)^2(n-2)} \geq \alpha$. In this case the $W_{2,n}(v)$ functions are negative by theorem 13,

start at $\frac{n}{2}(\frac{1}{n^2-1} - \frac{1}{\alpha^2})$, are monotonic increasing, and approach zero asymptotically as $v \to \infty$. For the equations of group II with $n < 0$ there will be precisely one solution if

$$\sqrt[4]{n(n+1)^2(n+2)} \geq \alpha \quad \text{and} \quad \frac{\delta}{\alpha} \geq \frac{n}{2}\left(\frac{1}{\alpha^2} - \frac{1}{n^2-1}\right) ; \qquad (10.3)$$

these solutions will be ordered as in (10.2). If the first but not the second of the inequalities in (10.3) is satisfied there will be no solution.

Theorem 12 is not likely to be useful in practice. As a consequence of this theorem we know that for $\alpha \geq \sqrt[4]{n(n+1)^2(n+2)}$, the functions $W_{2,n}(v)$ are positive, start at $\frac{n}{2}(\frac{1}{(n-1)^2} - \frac{1}{\alpha^2})$, are monotonic decreasing, and approach zero asymptotically as $v \to \infty$. For the equations of group I with $n > 0$ there will be precisely one solution if both

$$\alpha \geq \sqrt[4]{n(n+1)^2(n+2)} \quad \text{and} \quad \frac{\delta}{\alpha} \leq \frac{n}{2}\left(\frac{1}{n^2-1} - \frac{1}{\alpha^2}\right) ; \qquad (10.4)$$

these solutions will be ordered as $-- \; v_3 < v_2 < v_1 < v_0 < --$.
If the first but not the second of the inequalities in (10.4) is satisfied there will be no solution.

It follows from lemma 21 that $W_{2,n}(v)$ will be negative for all $v \geq 0$ if $\alpha \leq \sqrt{n(n-1)}$. Hence for an $n > 0$ such that $\alpha \leq \sqrt{n(n-1)}$ there will be no solution for the first equation in group I. Likewise for an $n < 0$ such that $\alpha \geq \sqrt{n(n-1)}$ there will be no solution for the second equation in group II.

This brings us to the end of the W-function theorems. We now proceed to fill in some of the gaps using semi-empirical methods. Figure 14 shows some typical graphs of the function $W_{2,1}(v)$. For all values of α the function $W_{2,1}$ starts at plus infinity; for $\alpha > 1$ the function approaches zero from above as $v \to \infty$ whereas for $\alpha \leq 1$ the function approaches zero from below as $v \to \infty$. For α in the neighborhood of 1.02 the function appears to have both a minimum and a maximum. For somewhat larger α the function seems to be monotonic decreasing. Finally for the case $\alpha \leq 1$ (the most important in practice), $W_{2,1}$ appears to have but a single

minimum. Thus for $\alpha \leq 1$, the $n = 1$ equation of group I has a single solution which is smaller than the $n = 0$ solution, and the $n = -1$ equation of group II has two solutions if δ/α is sufficiently small but otherwise no solution.

Figure 15 shows some typical graphs of the function $W_{2,2}(v)$. This function starts out at $(1/3 - 1/\alpha^2)$ which is positive for $\alpha > \sqrt{3}$ and negative for $\alpha < \sqrt{3}$. As v becomes infinite, $W_{2,2}(v)$ approaches zero from above for $\alpha > 2$ and from below for $\alpha \leq 2$. For sufficiently large α the function is of course monotonic decreasing; for α in the neighborhood of 2.02 it appears to have both a maximum and a minimum; for $\alpha \leq 2$ but somewhat greater than one it seems to have a single minimum; for $\alpha = 1$ (and hence by the concluding paragraph of section 9 for $\alpha \leq 1$) the function $W_{2,2}$ appears to be negative and monotonic increasing. One would say, therefore, that for $\alpha \leq \sqrt{3}$ the $n = 2$ equqtion of group I has no solution, whereas for $\alpha \leq 1$ the $n = -2$ equation of group II has a solution if $\frac{\delta}{\alpha} < \frac{1}{\alpha^2} \quad \frac{1}{3}$ and otherwise it has no solution.

The inbetween cases for $W_{2,3}$ are indicated in Fig. 16.

The results concerning the number of solutions that exist can be recapitulated as follows:

I. $n \geq 0$

A. Solutions associated with $W_{1,n}$ .... None

B. Solutions associated with $W_{2,n}$

$\alpha \leq \sqrt{n(n-1)}$; $n > 0$, ... None

$\alpha \geq \sqrt[4]{n(n+1)^2(n+2)}$ and $\frac{\delta}{\alpha} \leq \frac{n}{2}\left(\frac{1}{n^2-1} - \frac{1}{\alpha^2}\right)$..... One

$\alpha \geq \sqrt[4]{n(n+1)(n+2)}$ and $\frac{\delta}{\alpha} > \frac{n}{2}\left(\frac{1}{n^2-1} - \frac{1}{\alpha^2}\right)$..... None

$\sqrt{n(n-1)} < \alpha < \sqrt[4]{n(n+1)^2(n+2)}$ not covered by theory (See III below)

II. $n \leq 0$

A. Solutions associated with $W_{1,|n|}$ ....One

B. Solutions associated with $W_{2,|n|}$

$\alpha \leq \sqrt[4]{n(n+1)^2(n+2)}$ and $\dfrac{\delta}{\alpha} \geq \dfrac{n}{2}\left(\dfrac{1}{\alpha^2} - \dfrac{1}{n^2-1}\right)$...One

$\alpha \leq \sqrt[4]{n(n+1)^2(n+2)}$ and $\dfrac{\delta}{\alpha} < \dfrac{n}{2}\left(\dfrac{1}{\alpha^2} - \dfrac{1}{n^2-1}\right)$ ... None

$\alpha \leq \sqrt{n(n-1)}$, $n < 0$, ... None

$\sqrt{n(n-1)} \geq \alpha \geq \sqrt[4]{n(n+1)^2(n+2)}$ not covered by theory (See III below)

III. Cases examined graphically

$n = 1$, α about 1.02.................... Three

α a little greater than 1.02...... One

$\alpha \leq 1$ .... One

$n = -1$, $\dfrac{\delta}{\alpha}$ sufficiently small........ Three

$n = 2$, α about 2.02.................... Three

α a little greater than 2.02...... One

$\sqrt{3} < \alpha \leq 2$ .... One

$\alpha \leq \sqrt{3}$.................. None

$n = -2$, $\dfrac{\varepsilon}{\alpha} < \dfrac{1}{\alpha^2} - \dfrac{1}{3}$.......... One

otherwise .... None

Fig. 14

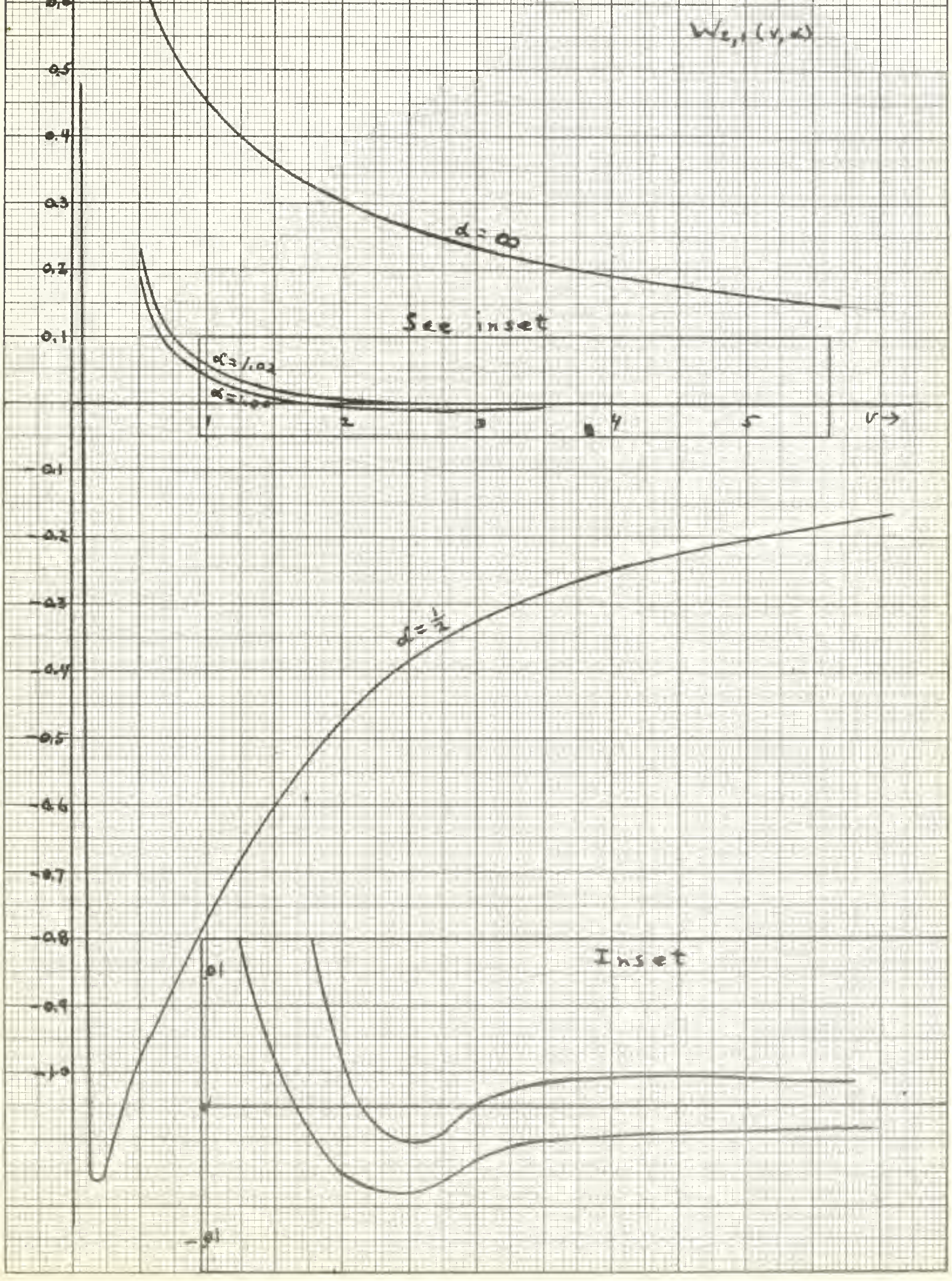

$W_{2,1}(v,\alpha)$
0.6
0.5
0.4
0.3
0.2
0.1
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1.0
$\alpha = \infty$
See inset
$\alpha = 1.02$
$\alpha = 1.00$
1
2
3
4
5
$v \rightarrow$
$\alpha = \frac{1}{2}$
Inset
.01
-.01

FIG. 15

TYPICAL GRAPHS OF $W_{2,2}(u)$

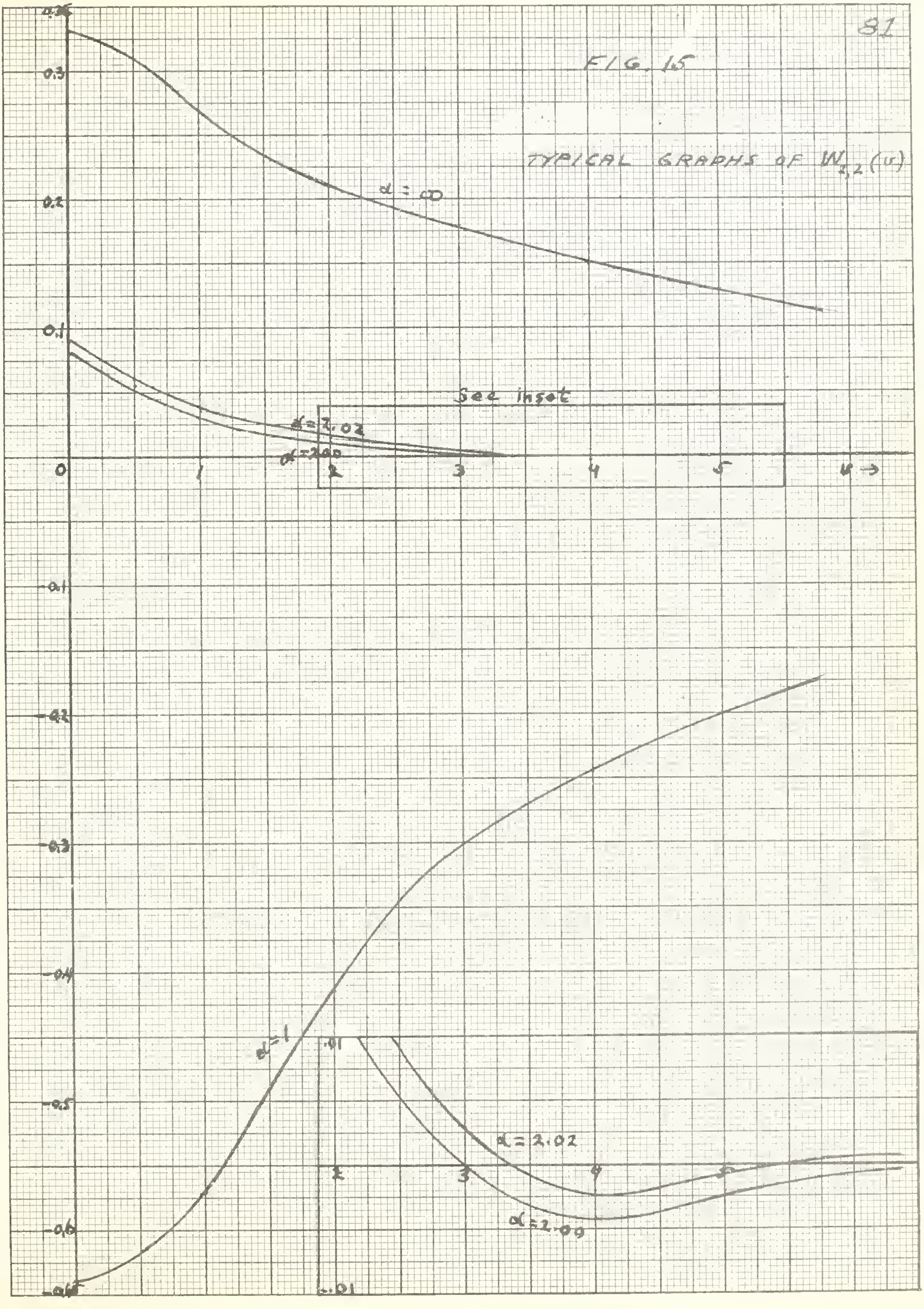
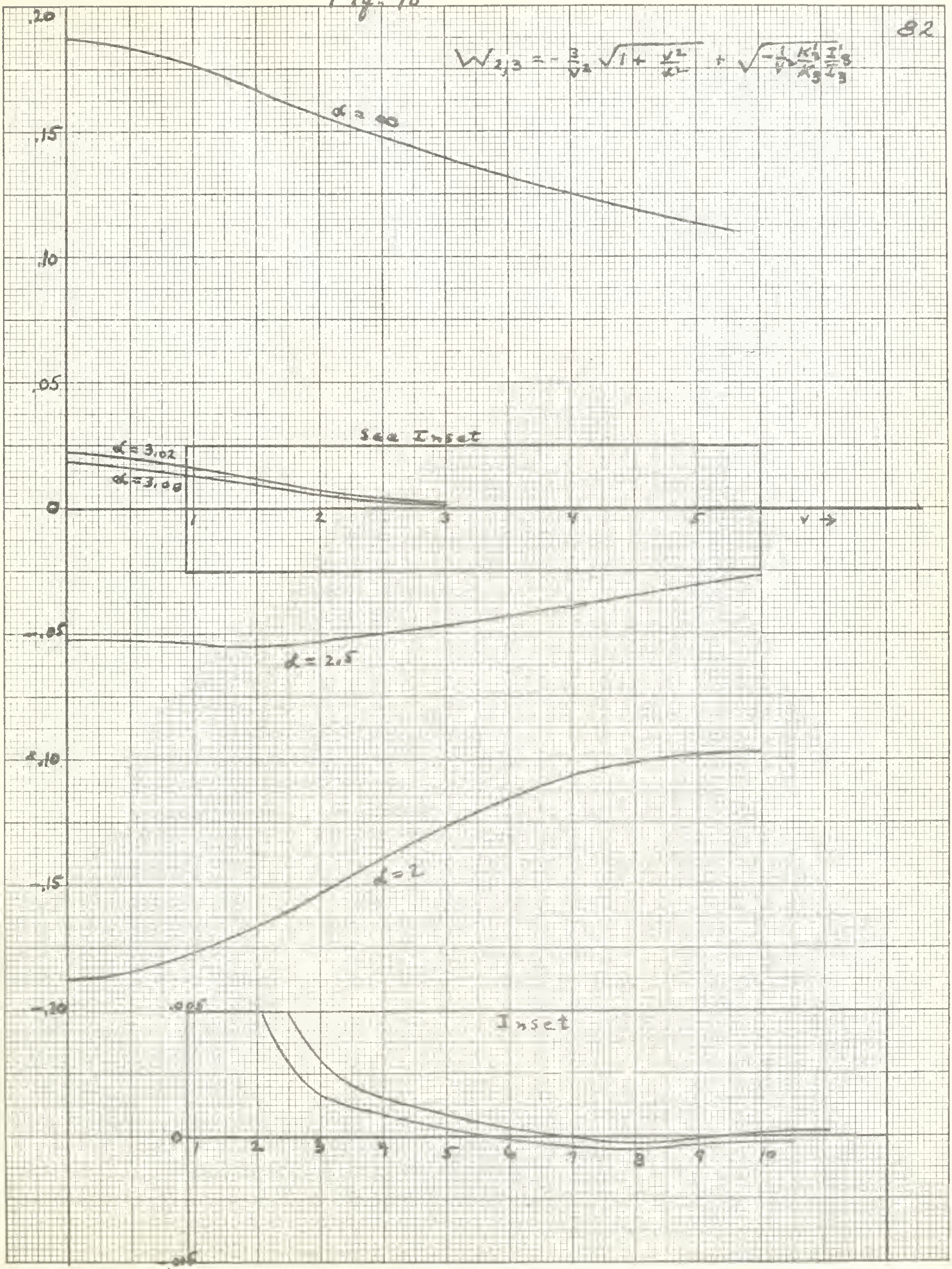


Fig. 16



$W_{2,3} = -\frac{3}{v^2}\sqrt{1+\frac{v^2}{\kappa^2}} + \sqrt{-\frac{1}{v^2}\frac{K_3'}{K_3}\frac{I_3'}{I_3}}$
.20
.15
.10
.05
0
-.05
-.10
-.15
-.20
$\alpha = \infty$
See Inset
$\alpha = 3.02$
$\alpha = 3.00$
1
2
3
4
5
$v \rightarrow$
$\alpha = 2.5$
$\alpha = 2$
Inset
.005
0
-.005
1
2
3
4
5
6
7
8
9
10